The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra and Linear Algebra
September 2023

Algebra

In this section, you can use any theorem from group theory, commutative algebra, Galois theory, etc. without proof as long as you state it clearly.

1. (10 points) (a) Prove that any module over $\mathbb{Z}[i]$ is a direct sum of a free module and a torsion module. Is the same true for modules over $\mathbb{Z}[\sqrt{-5}]$?
   (b) Let $I = (3 + 2i)$ be the ideal in $\mathbb{Z}[i]$ generated by the element $3 + 2i$. Describe the quotient $\mathbb{Z}[i]/I$.

2. (12 points) Let $p(x) = x^3 - x + 1$, and consider it as a polynomial over $\mathbb{F}_3$.
   (a) Describe the ring $R := \mathbb{F}_3[x]/(p(x))$.
   (b) Find the group of automorphisms of the ring R that leave $\mathbb{F}_3$ fixed.
   (c) Is the polynomial $x^3 + 9x^2 - 7x + 22$ reducible over $\mathbb{Q}$?
   (d) Prove that in a finite field, every element is a sum of two squares.
   Hint: this is a slightly harder question, unrelated to the previous parts of this problem.

3. (8 points) (a) Prove that any group of order 20 is solvable.
   (b) Find two non-abelian groups of order 20 not isomorphic to each other.

Linear Algebra

4. (10 points) (a) [4 points] Find a lower triangular matrix L such that $LL^T = A$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 5 & 2 & 2 \\ 0 & 2 & 2 & 1 \\ 2 & 2 & 1 & 6 \end{pmatrix}$$

   (b) [3 points] Compute $\det(L)$.
   (c) [3 points] Find the volume in $\mathbb{R}^4$ of the set $S_A = \{x \in \mathbb{R}^4 : x^T Ax \leq 1\}$.
   You can use the fact that the volume in $\mathbb{R}^4$ of the set $S_I = \{x \in \mathbb{R}^4 : x^T x \leq 1\}$ is $\frac{1}{2}\pi^2$.

5. (8 points) Let P_n be the $n+1$-dimensional space of polynomials of degree n with real coefficients, and let $\langle \cdot, \cdot \rangle$ be the inner product defined as

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

(a) [4 points] Find an orthogonal basis $\{u_0, u_1, u_2\}$ for P_2 such that $u_j \in P_j$, and $u_j(1) > 0$.
(b) [2 points] Using the basis in part (a), express the operator $F[p] := \int_{-1}^{1} p(x)dx$ acting on P_2 as a 1×3 matrix.
(c) [2 points] Using the basis in part (a), express the derivative operator $D[p] := \frac{d}{dx}p(x)$ as a 3×3 matrix.

6. (12 points) Recall that an orthogonal projection matrix is a matrix P that satisfies

$$P^2 = P, \quad P = P^T.$$

Suppose P is an $n \times n$ projection matrix with $\text{rank}(P) = k$. In the following, I_m denotes the $m \times m$ identity matrix.
(a) [4 points] List all the eigenvalues of P, including multiplicity. Be sure to justify your reasoning.
(b) [4 points] Show that $P = AA^T$ for some $n \times k$ matrix A such that $A^TA = I_k$.
(c) [4 points] Suppose P_1, P_2 are two $n \times n$ projection matrices with rank k. Show there exists an $n \times n$ orthonormal matrix U (i.e. such that $U^TU = I_n$, $UU^T = I_n$) such that $P_2 = UP_1U^T$.