## Course Outline for Mathematics 257/316 (3 credits) Term 1, Sept.-Dec., 2017

### Partial Differential Equations

**Prerequisites:** One of Math 215, 255, 265.

**Credit:** 3 Credits. Credit only given for one of Math 256, 257, 316.

**Instructor:** Anthony Peirce, **Office:** Mathematics Building 108

**Home Page:** [http://www.math.ubc.ca/~peirce](http://www.math.ubc.ca/~peirce)

**Office Hours:** Monday: 10-11 am, Wed: 3-3:55 pm, Fri: 10-11 am.

**Assessment:** The final grades will be based on homework (10%) (including EXCEL/MATLAB projects), two in class midterm exams (40%) and one final exam (50%). **Assignments are to be submitted in hard-copy from at the designated class – no late assignments can be accepted.** There will be no make-up midterms. A student must get at least 35% on the final exam to pass this course.

**Test Dates:** Wednesday, October 18th, Wednesday, November 15th.

**Text:** *Elementary Differential Equations and Boundary Value Problems (10th Ed),* W.E. Boyce & R.C. DiPrima (John Wiley & Sons) 2012

**Other References:**

### Topics

<table>
<thead>
<tr>
<th>Approx Time</th>
<th></th>
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<tbody>
<tr>
<td><strong>1.</strong> Review of techniques to solve ODEs</td>
<td>1 hr</td>
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<tr>
<td><strong>2.</strong> Series Solutions of variable coefficient ODEs (Chapter 5)</td>
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<tr>
<td>a. Series solutions at ordinary points (5.1-5.3)</td>
<td>3 hrs</td>
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<tr>
<td>b. Regular singular points (5.4-5.7, 5.8 briefly)</td>
<td>4 hrs</td>
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<tr>
<td><strong>3.</strong> Introduction to Partial differential equations (Chapter 10)</td>
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<tr>
<td>The heat equation (10.5), the wave equation (10.7), Laplace’s equation (10.8)</td>
<td>2 hrs</td>
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<tr>
<td><strong>4.</strong> Introduction to numerical methods for PDEs using spread sheets</td>
<td>3 hrs</td>
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<tr>
<td>a. First and second derivative approximations using finite differences - errors</td>
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<tr>
<td>b. Explicit finite difference schemes for the heat equation</td>
<td></td>
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<tr>
<td>• Stability and derivative boundary conditions</td>
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<tr>
<td>c. Explicit finite difference schemes for the wave equation</td>
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<tr>
<td>d. Finite difference approximation of Laplace’s Equation – iterative methods</td>
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<tr>
<td><strong>5.</strong> Fourier Series and Separation of Variables (Chapter 10)</td>
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<tr>
<td>a. The heat equation and Fourier Series (10.1-10.6)</td>
<td>9 hrs</td>
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<tr>
<td>b. The wave equation (10.7)</td>
<td>3 hrs</td>
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<tr>
<td>c. Laplace’s equation (10.8)</td>
<td>5 hrs</td>
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<tr>
<td><strong>6.</strong> Boundary Value Problems and Sturm-Liouville Theory (Chapter 11)</td>
<td></td>
</tr>
<tr>
<td>a. Eigenfunctions and eigenvalues (11.1)</td>
<td>1 hr</td>
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<tr>
<td>b. Sturm-Liouville boundary value problems (11.2)</td>
<td>1 hr</td>
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<tr>
<td>c. Nonhomogeneous boundary value problems (11.3)</td>
<td>2 hrs</td>
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<tr>
<td>Tests</td>
<td>2 hrs</td>
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<tr>
<td></td>
<td>36 hrs</td>
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**Math 257-316 PDE  Formula sheet - final exam**

### Trigonometric and Hyperbolic Function identities

\[
\begin{align*}
\sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \sin \beta \cos \alpha \\
\cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \beta \sin \alpha. \\
\sinh(\alpha \pm \beta) &= \sinh \alpha \cosh \beta \pm \sinh \beta \cosh \alpha. \\
\cosh(\alpha \pm \beta) &= \cosh \alpha \cosh \beta \pm \sinh \alpha \sinh \beta.
\end{align*}
\]

\[
\begin{align*}
\sinh^2 t + \cosh^2 t &= 1 \\
\sin^2 t + \cos^2 t &= 1 \\
\sinh^2 t &= \frac{1}{2} (1 - \cosh(2t)) \\
\cosh^2 t - \sinh^2 t &= 1 \\
\sinh^2 t &= \frac{1}{2} (\cosh(2t) - 1)
\end{align*}
\]

### Basic linear ODE's with real coefficients

<table>
<thead>
<tr>
<th>ODE</th>
<th>constant coefficients</th>
<th>Euler eq</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ay'' + by' + cy = 0 )</td>
<td>( ax^r y'' + bx y' + cy = 0 )</td>
<td>( ar(r-1) + br + c = 0 )</td>
</tr>
<tr>
<td>Indicial eq.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( r_1 \neq r_2 )</td>
<td>( y = Ae^{r_1x} + Be^{r_2x} )</td>
<td>( y = Ar^x + Br^2 )</td>
</tr>
<tr>
<td>( r_1 = r_2 = r )</td>
<td>( y = A e^{rx} + Bxe^{rx} )</td>
<td>( y = Ax^r + Bx \ln</td>
</tr>
<tr>
<td>( r = \lambda \pm i\mu )</td>
<td>( e^{\lambda x}[A \cos(\mu x) + B \sin(\mu x)] )</td>
<td>( x^\lambda[A \cos(\mu \ln</td>
</tr>
</tbody>
</table>

### Series solutions for \( y'' + p(x) y' + q(x) y = 0 \) around \( x = x_0 \).

**Ordinary point** \( x_0 \): Two linearly independent solutions of the form:

\[
y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n
\]

**Regular singular point** \( x_0 \): Rearrange (\*) as:

\[
(x-x_0)^2 y'' + [(x-x_0)p(x)] (x-x_0)y' + [(x-x_0)^2 q(x)] y = 0
\]

If \( r_1 > r_2 \) are roots of the indicial equation: \( r(r-1) + br + c = 0 \) where

\[
b = \lim_{x \to x_0} (x-x_0)p(x) \quad \text{and} \quad c = \lim_{x \to x_0} (x-x_0)^2 q(x)
\]

Then a solution of (\*) is

\[
y_1(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r_1}
\]

where \( a_0 = 1 \).

The second linearly independent solution \( y_2 \) is of the form:

**Case 1:** If \( r_1 - r_2 \) is neither 0 nor a positive integer:

\[
y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}
\]

where \( b_0 = 1 \).

**Case 2:** If \( r_1 - r_2 = 0 \):

\[
y_2(x) = y_1(x) \ln(x-x_0) + \sum_{n=1}^{\infty} b_n (x-x_0)^{n+r_2}
\]

for some \( b_1, b_2 \ldots \)

**Case 3:** If \( r_1 - r_2 \) is a positive integer:

\[
y_2(x) = a y_1(x) \ln(x-x_0) + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}
\]

where \( b_0 = 1 \).

### Fourier, sine and cosine series

Let \( f(x) \) be defined in \([-L, L]\) then its Fourier series \( Ff(x) \) is a \( 2L \)-periodic function on \( \mathbb{R} \):

\[
Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}
\]

where

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \\
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx
\]

**Theorem (Pointwise convergence)** If \( f(x) \) and \( f'(x) \) are piecewise continuous, then \( Ff(x) \) converges for every \( x \) to \( \frac{1}{2} [f(x-) + f(x+)] \).

**Parseval's identity**

\[
\frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)
\]

For \( f(x) \) defined in \([0, L]\), its cosine and sine series are

\[
Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx,
\]

\[
Sf(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx.
\]

### Sturm-Liouville Eigenvalue Problems

**ODE:** \( p(x)y'' - q(x)y + \lambda r(x)y = 0 \), \( a < x < b \).

**BC:** \( \alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0 \).

**Hypothesis:** \( p, p', q, r \) continuous on \([a, b]\). \( p(x) > 0 \) and \( r(x) > 0 \) for \( x \in [a, b] \), \( \alpha_1^2 + \alpha_2^2 > 0, \beta_1^2 + \beta_2^2 > 0 \).

**Properties** (1) The differential operator \( L \) is symmetric in the sense that \( (f, Lg) = (Lf, g) \) for all \( f, g \) satisfying the BC, where \( (f, g) = \int_{a}^{b} f(x)g(x) \, dx \). (2) All eigenvalues are real and can be ordered as \( \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \) with \( \lambda_n \to \infty \) as \( n \to \infty \), and each eigenvalue admits a unique (up to a scalar factor) eigenfunction \( \phi_n \).

(3) **Orthogonality:** \( \langle \phi_m, r \phi_n \rangle = \int_{a}^{b} \phi_m(x) \phi_n(x) r(x) \, dx = 0 \) if \( m \neq n \).

(4) **Expansion:** If \( f(x) : [a, b] \to \mathbb{R} \) is square integrable, then

\[
f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad a < x < b, \quad c_n = \frac{1}{\phi_n(a) \phi_n(b)} \int_{a}^{b} f(x) \phi_n(x) r(x) \, dx, \quad n = 1, 2, \ldots
\]