Let $Y$ be a topological space and let $A$ be a set. Let $Y^A = \{ f : A \to Y \}$ be the product space $\prod_{\alpha \in A} Y$ with the product topology.

(a) The product topology on $Y^A$ is the weakest topology such that . . . ?

(b) Describe a neighbourhood base for a point $f \in Y^A$.

(c) Show that pointwise convergence, $f_n(\alpha) \to f(\alpha)$ for each $\alpha \in A$, implies $f_n \to f$.

(d) Does every sequence $\{f_n\}$ with $f_n \in \{0,1\}^{[0,1)}$ have a convergent subsequence? (Yes/No plus very brief comment in either case).

2. Let $\mathcal{X}$ be a normed vector space over the complex numbers and let $\mathcal{X}^*$ be the space of continuous linear functionals on $\mathcal{X}$.

(a) Define the norm $\|f\|$ of $f \in \mathcal{X}^*$.

(b) State the complex version of the Hahn Banach theorem.

(c) Let $x_0 \in \mathcal{X}$. Show that there is a linear functional $f \in \mathcal{X}^*$ such that $f(x_0) = \|x_0\|$ and $\|f\| = 1$.

(d) Suppose that $x_n \to x$ weakly. Prove that $\|x\| \leq \lim \inf \|x_n\|$.

(e) Suppose that $\mathcal{X}$ is a Hilbert space, that $x_n \to x$ weakly and $\|x\| = \lim \|x_n\|$. Prove that $x_n \to x$ in norm.

(f) Is it possible for $x_n \to x$ weakly and $\|x\| < \lim \inf \|x_n\|$? Hint: Bessel inequality.

3. (a) Are continuous functions dense in $L^\infty([0,1], dx)$? (Yes/No plus brief explanation in either case).
(b) Define the term complete orthonormal set (orthonormal basis) in the context of a separable Hilbert space.

(c) Prove that if \( f \perp D \) where \( D \) is a dense subset of a Hilbert space, then \( f = 0 \).

(d) For \( k \in \mathbb{Z} \) and \( x \in [0, 2\pi] \), let \( e_k(x) = (2\pi)^{-1/2} e^{ikx} \). You may assume these functions are an orthonormal set in \( L^2([0,2\pi]) \) and that continuous functions compactly supported in \((0,2\pi)\) are dense in \( L^2([0,2\pi]) \). Prove that \( \{e_k\} \) is a complete orthonormal set in \( L^2([0,2\pi]) \).

4. Let \( \mathcal{X} \) be a Banach space, let \( \{T_n\} \in L(\mathcal{X}, \mathcal{X}) \) be a sequence of continuous linear operators on \( \mathcal{X} \).

(a) There are at least three notions of convergence for the sequence \( T_n \). What are they?

(b) Suppose, \( \forall x \in \mathcal{X}, \forall f \in \mathcal{X}^* \), that \( f(T_n x) \to f(T x) \) where \( T \) is a linear operator. Show that \( T \in L(\mathcal{X}, \mathcal{X}) \).

5. Let \( T \in L(\mathcal{X}, \mathcal{X}) \), where \( \mathcal{X} \) is a Banach space.

(a) Define the resolvent set \( \rho(T) \) and the resolvent \( R_\lambda \) of \( T \).

(b) Prove that

\[
T = \frac{1}{2\pi i} \oint_\Gamma R_\lambda \lambda d\lambda,
\]

where \( \Gamma \) is the oriented boundary of an open disk \( D \supset \sigma(T) \).