All 5 questions carry equal credit. No calculators, books or notes allowed.
(1) (a) For a measure space \((X, \mathcal{M}, \mu)\) and \(p \in [1, \infty)\) define: (i) \(\|f\|_p\); (ii) \(L^p\); (iii) convergence in \(L^p\).

(b) State the Hölder and Minkowski inequalities. (You do not need to say when equality holds).

(c) Let \(p, q \in (1, \infty)\) satisfy \(1/p + 1/q = 1\). Show that if \(f, f_1, f_2, \ldots \in L^p\) satisfy \(f_n \to f\) in \(L^p\) and \(g, g_1, g_2, \ldots \in L^q\) satisfy \(g_n \to g\) in \(L^q\), then \(f_n g_n \to fg\) in \(L^1\). (Here \(fg\) denotes the pointwise product).

(d) For some \(p, q \in (1, \infty)\) with \(1/p + 1/q \neq 1\) give an example to show that the implication in (c) need not hold.

(2) Let \(\mu, \nu, \lambda\) be \(\sigma\)-finite positive measures on \((X, \mathcal{M})\).

(a) Show that \(\mu \ll \mu + \nu\).

(b) Show that if \(\nu \ll \mu\) and \(\lambda \ll \mu\) then \(\nu + \lambda \ll \mu\) and

\[
\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \quad \mu\text{-a.e.}
\]

(c) Show that if \(\lambda \ll \nu \ll \mu\) then \(\lambda \ll \mu\) and

\[
\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \frac{d\nu}{d\mu} \quad \mu\text{-a.e.}
\]

(d) Show that if \(\lambda \ll \mu\) and \(\lambda \ll \nu\) then \(\lambda \ll \mu + \nu\); find and prove a formula for \(\frac{d\lambda}{d(\mu + \nu)}\) in terms of (only) \(\frac{d\lambda}{d\mu}\) and \(\frac{d\lambda}{d\nu}\), assuming that \(\frac{d\lambda}{d\mu}, \frac{d\lambda}{d\nu} \in (0, \infty)\).

(3) (a) State: (i) the monotone convergence theorem; (ii) Fatou’s lemma; (iii) the dominated convergence theorem.

(b) Assuming (ii), prove (iii).

(c) Evaluate \(\lim_{n \to \infty} \int_0^\infty \frac{\sin(x/n)}{x + x^2} \, dx\), justifying your answer.
(4) Let $m$ denote Lebesgue measure on $\mathbb{R}^2$.

(a) Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is Borel-measurable then

$$m\{(x, f(x)) : x \in \mathbb{R}\} = 0.$$ 

(b) Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is Borel-measurable then

$$m\{(x + f(x), x - f(x)) : x \in \mathbb{R}\} = 0.$$ 

Hint: apply a transformation of $\mathbb{R}^2$.

(c) Show that if $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are increasing then

$$m\{(f(t), g(t)) : t \in \mathbb{R}\} = 0.$$ 

Hint: consider the intersection of the set with the line $\{(x, y) : x + y = a\}$.

(5) Let $f, f_1, f_2, \ldots$ be measurable real functions on $(X, \mathcal{M}, \mu)$. For $A \subset X$, recall that “$f_n \rightarrow f$ uniformly on $A$” means that for every $\epsilon > 0$ there exists $N$ such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } n \geq N \text{ and } x \in A.$$ 

We say that “$f_n \rightarrow f$ almost uniformly” if for every $\delta > 0$ there exists $A \in \mathcal{M}$ with $\mu(A^C) < \delta$ such that $f_n \rightarrow f$ uniformly on $A$.

(a) Show that if $f_n \rightarrow f$ almost uniformly then $f_n \rightarrow f$ almost everywhere.

(b) Suppose $\mu(X) < \infty$. Show that if $f_n \rightarrow f$ almost everywhere then $f_n \rightarrow f$ almost uniformly.

(Hints: Let $E(\epsilon, N)$ be the set of $x$ such that $|f_n(x) - f(x)| > \epsilon$ for some $n \geq N$. Show that $\lim_{N \rightarrow \infty} \mu(E(\epsilon, N)) = 0$. Then choose $N_k$ such that $\mu(E(1/k, N_k)) \leq \delta 2^{-k}$.)

(c) Give an example to show that if $\mu(X) = \infty$ then the implication in (b) need not hold.