1. Consider the ODE problem for $u(x)$:

$$\begin{cases} \quad Lu := (p(x)u')' + q(x)u = f(x), & 1 < x < 2 \\ u(1) = 2, & u(2) = 5 \end{cases}$$

($p(x)$, $q(x)$, and $f(x)$ are given functions).

(a) (2 pts.) Write down the problem satisfied by the Green’s function $G_x(y) = G(x; y)$ for problem (1).

(b) (2 pts.) Express the solution $u(x)$ of (1) in terms of the Green’s function $G_x(y)$.

(c) (3 pts.) If $p(x) = 1/x^2$ and $q(x) = 2/x^4$, find the Green’s function $G(x; y)$ for (1).

(d) (3 pts.) Again with $p(x) = 1/x^2$ and $q(x) = 2/x^4$, find the solvability condition on $f(x)$ for (1) if the boundary conditions are changed to $u'(1) = 2$, $u(2) = 5$. 
2. The free-space Green’s function for $\Delta$ in the plane $\mathbb{R}^2$ is $G_{\mathbb{R}^2}(y) = A \ln |y - x|$.

(a) (3 pts.) Derive the value of the constant $A$.

(b) (5 pts.) Use the method of images to find the Green’s function for the following boundary-value problem for Laplace’s equation in the quarter-plane,

\[
\begin{align*}
\Delta u &= 0, \quad x_1 > 0, \quad x_2 > 0 \\
u(0, x_2) &= 0, \quad u(x_1, 0) = g(x_1)
\end{align*}
\]

and find the solution $u(x_1, x_2)$.

(c) (2 pts.) Prove that problem (2) cannot have more than one solution $u(x)$ which decays at infinity (i.e. $\lim_{|x| \to \infty} u(x) = 0$).
3. Let $D$ be a bounded domain in $\mathbb{R}^n$, and let $\{\phi_j(x)\}_{j=1}^{\infty}$ be a complete, orthonormal set of eigenfunctions for $-\Delta$ on $D$ with zero (Dirichlet) BCs: $-\Delta \phi_j(x) = \lambda_j \phi_j(x)$, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$.

(a) (4 pts.) Write down the problem satisfied by the Green’s function $G(y, \tau; x, t)$ for the following problem for a heat-type equation

$$\begin{cases}
  & u_t = \Delta u + u \quad x \in D, \ t > 0 \\
  & u(x, 0) = u_0(x) \quad x \in D \\
  & u(x, t) \equiv 0 \quad x \in \partial D
\end{cases}$$

and express the solution $u(x, t)$ in terms of the Green’s function.

(b) (4 pts.) Find the Green’s function as an eigenfunction expansion.

(c) (2 pts.) Under what condition will typical solutions grow with time?
4. Consider the variational problem

$$\min_{u \in C^4([0,1])} \int_0^1 \left\{ \frac{1}{2}(u''(x))^2 + \frac{1}{2}(u(x))^2 - e^x u(x) \right\} dx.$$ 

(a) (5 pts.) Determine the problem (Euler-Lagrange equation plus BCs) that a minimizer $u(x)$ solves.

(b) (5 pts.) Find an approximate minimizer, using a Rayleigh-Ritz approach, with two trial functions $v_1(x) = e^x$, $v_2(x) = e^{-x}$. 
5. Let $D$ denote the half unit disk:

$$D = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1, \ x_2 > 0 \},$$

and let $\lambda_1$ denote the first Dirichlet (zero BCs) eigenvalue of $-\Delta$ on $D$.

(a) (4 pts.) Find the best upper- and lower-bounds you can for $\lambda_1$ by comparing $D$ with appropriate rectangles.

(b) (3 pts.) Write the variational principle for $\lambda_1$, and use it with trial function $v(x) = x_2(1 - x_1^2 - x_2^2)$ to find an upper bound for $\lambda_1$ (you may wish to compute in polar coordinates).

(c) (3 pts.) Find the exact value of $\lambda_1$ in terms of the first positive root of the Bessel function $J_1(z)$ (which is the solution of the ODE $z^2J''(z) + zJ'(z) + (z^2 - 1)J(z) = 0$ which is finite at $z = 0$).