1. Consider the problem
\[
\begin{aligned}
&u'' + u = f(x), \quad 0 < x < \pi/2 \\
&u(0) = 0, \quad u(\pi/2) = 0
\end{aligned}
\] (1)

(a) (4 pts.) Write down the problem that the Green’s function $G(x; y)$ for problem (1) should solve.

(b) (7 pts.) Find the Green’s function $G(x; y)$ for problem (1), and express the solution $u(x)$ in terms of it.

(c) (6 pts.) Find the solvability condition on $f$ if the boundary conditions are changed to $u(0) + u'(\pi/2) = 0, \quad u(\pi/2) = 0.$
2. Let $D$ be a bounded (and smooth, and open) region in $\mathbb{R}^n$, and consider the following Poisson boundary-value problem:

$$
\begin{align*}
\begin{cases}
\Delta u = f(x) & \text{in } D \\
u = g(x) & \text{on } \partial D
\end{cases}
\end{align*}
$$

(for given smooth functions $f$ and $g$ on $D$ and $\partial D$ respectively).

(a) (6 pts.) Write down the problem that the Green’s function $G(x; y)$ for problem (2) should solve, and express the solution $u(x)$ in terms of $G$.

(b) (6 pts.) Derive an expression for the Green’s function $G(x; y)$ in terms of an orthonormal family of eigenfunctions $\phi_j(x)$, $j = 1, 2, 3, \ldots$, of $\Delta$ on $D$ (with zero boundary conditions), and the corresponding eigenvalues $\lambda_j$.

(c) (5 pts.) Suppose that $f(x) \equiv 0$, and $g(x) \geq 0$ with $g(x_0) > 0$ for some $x_0 \in \partial D$. Show that the solution $u$ of (2) satisfies $u(x) > 0$ for $x \in D$. (Hint: maximum principle).
3. (17 pts.) Consider the following problem for the wave equation on the half-line with a Neumann boundary condition:

\[
\begin{cases}
    u_{tt} = u_{xx} & x > 0, \ t > 0 \\
    u_x(0, t) = 0, & u \to 0 \text{ as } x \to +\infty \\
    u(x, 0) = u_0(x), \quad u_t(x, 0) = 0
\end{cases}
\]

where \( u_0(x) \) is a smooth function tending to 0 as \( x \to +\infty \). Find the Green’s function for this problem, and use it find the solution \( u(x, t) \). (Hint: recall the Green’s function for the wave equation on the entire line is \( G_E(x, t; y, \tau) = \frac{1}{2}H(t - \tau - |y - x|) \) where \( H \) is the Heavyside step function.)
4. (a) (6 pts.) Derive (from first principles) the Euler-Lagrange equation, as well as
the boundary conditions, satisfied by solutions of this variational problem:

$$\min_{u \in C^2([0,1])} \int_0^1 F(u(x), u'(x), x) \, dx.$$ 

(b) (6 pts.) Find the minimizing function for the problem

$$\min_{u(0)=0, u(1)=1} \int_0^1 (|u'(x)|^2 + |u(x)|^2) \, dx$$

(c) (5 pts.) Write a variational problem for functions \(u(x), x \in [0,1]\) whose Euler-
Lagrange equation is \(u'' = -\sin(u)\). (Don’t worry about boundary conditions.)
( Remark: this is the “nonlinear pendulum equation” with \(x\) as time, and \(u\) as
angle of displacement.)
5. Let $D$ be a bounded domain in $\mathbb{R}^n$, $p(x) > 0$ a smooth function on $D$, and consider the (Dirichlet) eigenvalue problem
\[
\begin{cases}
-\nabla \cdot [p(x)\nabla \phi] = \lambda \phi & \text{in } D \\
\phi = 0 & \text{on } \partial D
\end{cases}
\]  
(3)

(a) (3 pts.) Explain how to get an upper bound for the first eigenvalue $\lambda_1$ of problem (3) using a smooth trial function $v(x)$ on $D$ with $v = 0$ on $\partial D$.

(b) Now suppose $D = D_1$ is the unit disk in $\mathbb{R}^2$, $D_1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$, and suppose $p(x) = 1 + |x|^2$.

i. (5 pts.) Find an upper bound on $\lambda_1$ using trial function $v(x) = 1 - |x|^2$.

(Hint: do the integrals in polar coordinates).

ii. (5 pts.) Find a lower bound on $\lambda_1$ by comparing $D_1$ with an appropriate square.

iii. (4 pts.) Explain how you might go about getting an upper bound for the second eigenvalue $\lambda_2$ by using two trial functions (for example, $v(x) = 1 - |x|^2$ and $w(x) = 1 - |x|^4$) – but do not try to do any computations.
6. (15 pts.) Let $D_1$ be the unit disk in $\mathbb{R}^2$. Use a Rayleigh-Ritz-type approach to find an approximate solution to the variational problem

$$\min_{u \in C^2(D_1), \, u(x) \equiv 1 \text{ on } \partial D_1} \int_{D_1} (|\nabla u|^2 + e^u) \, dx$$

by considering the one-parameter family of trial functions

$$u(x) = 1 + a(1 - |x|^2), \quad a \in \mathbb{R}.$$ 

Reduce the problem to an algebraic equation for $a$, but do not try to solve this equation. (Hint: again, polar coordinates might help).