1. Consider the following first order PDE

\[ u_t + u^3 u_x = 0, \quad t > 0, \quad -\infty < x < +\infty \]

with \( u(x, 0) = 1 \) when \( 0 < x < 1 \) and \( u(x, 0) = 0 \) otherwise.

(i) (15) Find the solution with expansion fan and shock.

(ii) (5) Locate the time \( t_B \) when the expansion fan hits the shock. Find the solution when \( t > t_B \).
2. Consider the following wave equation

\[ u_{tt} - 4u_{xx} = f(x,t), \quad 0 < x < +\infty, \quad t > 0 \]

\[ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad 0 < x < +\infty \]

\[ u(0,t) = h(t), \quad t > 0 \]

(i) (10) Find the general solution to the above wave equation when \( f = 0, \phi = 0 \) and \( \psi = 0 \)

(ii) (10) Find the solution to the above wave equation with

\[ f(x,t) = xt, \quad \phi(x) = 1, \quad \psi(x) = \sin x, \quad h(t) = e^t \]

(iii) (5) Use the energy method to show that the solution to the above wave equation is unique.

3. Consider the following diffusion equation

\[ u_t = u_{xx} + 2u_x + u, \quad 0 < x < 1, \quad t > 0 \]

\[ u(x,0) = \phi(x), \quad 0 < x < 1 \]

\[ u(0,t) = 0, \quad 2u_x(1,t) - u(1,t) = 0, \quad t > 0 \]

(i) (20) Use the method of separation of variables to find the general solution.

(ii) (5) Under what condition on the initial condition \( \phi \) is the solution obtained in (i) bounded? Justify your answer.
4. Consider the following Laplace equation $u_{xx} + u_{yy} = 0$ in the disk of radius $a$ defined by $D = \{(x, y) \mid x^2 + y^2 < a^2\}$, with $u(x, y) = 1 + x^2 + 3xy$ on the boundary of $D$: $x^2 + y^2 = a^2$.

(5) Without solving the problem explicitly, find the value of $u(0, 0)$, and the maximum and minimum values of $u$ in $D$. (5) Justify your answer.

5. Use the method of separation of variables to solve the following Laplace equation

$u_{xx} + u_{yy} = 0$ in $D = \{(x, y) \mid x^2 + y^2 > 4, \ x > 0, \ y > 0\}$,

$u_y(x, 0) = 0$ for $x > 0$, and $u(0, y) = 0$ for $y > 0$,

$u(x, y) = y^2$ on $x^2 + y^2 = 4, x > 0, y > 0$,

$u(x, y)$ is bounded.

Total Marks