Math 322, Fall Term 2011
Final Exam

Do not turn this page over until instructed

- On the top of each exam booklet write
  1. Your student ID;
  2. The serial number of this exam;
  3. The number of the booklet (if you use more than one).

- On the sign-in sheet write
  1. Your student ID;
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  3. Your name.

Instructions

- You will have 150 minutes for this exam.
- No books, notes or electronic devices.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you use a result from the lectures or the problem sets, quote it properly.
- Good luck!
1. \([8+6+6=20\text{pts}]\)
   (a) Define the sign of an element in \(S_n\) and the alternating group \(A_n\). Determine the sign of \(\sigma = (1\ 2\ 3\ 4\ 5)\) in \(S_5\) and the sign of a cycle of type \((3,4,5)\) in \(S_{12}\).
   
   (b) For \(n > 2\), prove that the centre of \(S_n\) is trivial.
   
   (c) Write an element in \(S_4\) of cycle type \((2,2)\) and determine the number of elements in its conjugacy class.

2. \([7+8+5=20\text{pts}]\)
   (a) Write down the class equation of a group and use it to show that a finite group of order \(p^n\), \(p\) a prime, has non-trivial centre.
   
   (b) Show that any group of order 45 is abelian and any group of order 33 is cyclic.
   
   (c) Classify the possible abelian groups of order 792.

3. \([8+8+6=22\text{pts}]\)
   (a) Prove that \(f(x) = x^4 + 2x + 2\) is irreducible over \(\mathbb{Q}\). Define a prime ideal and maximal ideal in a ring \(R\) and give an example of a non-zero prime ideal in \(\mathbb{Z}[x]\).
   
   (b) Is 13 a prime in \(\mathbb{Z}[i]\)? Justify your answer. Show that any ideal in \(\mathbb{Z}[i]\) contains a positive integer.
   
   (c) If \(f : R \to S\) is a ring homomorphism between commutative rings show that \(f^{-1}(P)\) is a prime ideal in \(R\) whenever \(P\) is a prime ideal in \(S\).

4. \([10+6+6=22\text{pts}]\)
   (a) Show that \(f(x) = x^3 + 2x + 2\) is irreducible in \(\mathbb{F}_3[x]\) (\(\mathbb{F}_3\) is the finite field with three elements) and use this fact to construct a field with 27 elements that contains \(\mathbb{F}_3\).
   
   (b) Consider the polynomial \(f(x) = (x^2 + 1)(x^2 - 2)\) over \(\mathbb{Q}\). Find a field extension of \(\mathbb{Q}\) where \(f(x)\) splits completely into linear factors. What is the degree of such a field extension?
   
   (c) Let \(E/F\) be a field extension. What does it mean to say that \(E/F\) is algebraic? Give a necessary and sufficient condition for an element \(\alpha\) in \(E\) to be algebraic over \(F\). Give an example of a complex number which is not real and is algebraic over \(\mathbb{Q}\).

5. State whether the following are true or false with full justification \([8x2=16\text{pts}]\)
   (a) If \(\tau\) is a cycle in \(S_n\) then \(\tau^2\) is also a cycle.
(b) Let $R = \mathbb{Z}[i]$ be the ring of Euclidean integers. In the polynomial ring $R[X]$ the prime elements are the same as irreducible elements.

(c) If a group of order 25 acts on a set $X$, then there is an element $x \in X$ such that its orbit has 8 elements.

(d) The number of conjugacy classes in $S_n$ is equal to the number of non-isomorphic abelian groups of order $p^n$, for any prime $p$.

(e) The group $S_6$ has a subgroup $H$ of order 360 whose normaliser is $S_6$.

(f) The polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ is irreducible in $\mathbb{Z}[x]$.

(g) In $\mathbb{Z}[x]$, every prime ideal is maximal.

(h) If $P$ is a $p$-Sylow subgroup and $Q$ is a $q$-Sylow subgroup for two distinct primes $p$ and $q$, and if $x \in P$, then no conjugate of $x$ lies in $Q$. 