1. [16 points] Determine whether the following statements are true or false (you have to include proofs/counterexamples):
   (a) The rings \( \mathbb{Z}/35\mathbb{Z} \) and \( \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \) are isomorphic.
   (b) The groups \( \mathbb{Z}/24\mathbb{Z} \) and \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \) are isomorphic.
   (c) If \( G \) is a cyclic group of order \( n \), and \( d|n \), then \( G \) has a subgroup of order \( d \).
   (d) The groups \( \mathbb{F}_p^\times \) and \( (\mathbb{Z}/p^2\mathbb{Z})^\times \) are isomorphic.

2. [13 points]
   (a) Let \( G \) be a group, and \( N \) – a normal subgroup of \( G \). Prove that if \( N \) contains an element \( g \), then \( N \) contains the entire conjugacy class of \( g \).
   (b) Let \( \sigma \) be the following element of \( S_4 \):
       \[ \sigma = (1 \ 2 \ 3 \ 4 \ 2 \ 1 \ 3 \ 4) \]
       Find the number of elements in the conjugacy class of \( \sigma \).
   (c) Prove that the permutation \( \sigma \) from Part (b) cannot be contained in any proper normal subgroup of \( S_4 \).

3. [8 points] Let \( M \) and \( N \) be normal subgroups of a group \( G \). Suppose that \( M \cap N = \{e\} \). Prove that for every \( m \in M \) and \( n \in N \),
   \[ mn = nm \]

4. [12 points] Find the set of units and the set of zero divisors in the ring \( R \), where:
   (a) \( R \) is the ring \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).
   (b) Let \( R \) is the quotient ring \( \mathbb{Z}[i]/I \), where \( I \) is the ideal
       \[ I = \{a + b\sqrt{7} \mid 6|a - b\} \]
       (Hint: find a convenient homomorphism from \( \mathbb{Z}[i] \) to \( \mathbb{Z}/6\mathbb{Z} \)).

5. [10 points] Let \( R = \mathbb{F}_5[x] \); let \( I = \langle x^2 + 1 \rangle \) be the ideal in \( R \) generated by the polynomial \( x^2 + 1 \), and let \( J = \langle x^3 + 2 \rangle \) be the ideal generated by the polynomial \( x^3 + 2 \). Prove that \( I + J \) is a principal ideal in \( R \), and find its generator.

6. [10 points]
   (a) Factor the element \( 5 \in \mathbb{Z}[i] \) as a product of irreducible elements.
   (b) Is \( \langle 5 \rangle \) a maximal ideal in \( \mathbb{Z}[i] \)?

7. [15 points]
   (a) Prove that the polynomial \( x^3 + 2x + 1 \) is irreducible in \( \mathbb{F}_3[x] \).
   (b) Let \( I \) be the ideal \( I = \langle x^3 + 2x + 1 \rangle \) in \( \mathbb{F}_3[x] \), and let \( R = \mathbb{F}_3[x]/I \). Let \( \alpha = x + I \in R \). Prove that the element \( 1 + \alpha \) has an inverse in \( R \).
   (c) Find \( (1 + \alpha)^{-1} \) (that is, find \( \gamma \in R \), such that \( \gamma(1 + \alpha) = 1 \)).

8. [8 points] Is \( \mathbb{Z}[\sqrt{-3}] \) a principal ideal domain?

9. [8 points] Let \( G \) be a group acting on a set \( X \). Suppose that the stabilizer \( G_x \) of a certain point \( x \in X \) is a proper normal subgroup of \( G \). Prove that every element of \( G_x \) fixes every point \( y \in O_x \).
Extra credit problems:

1. Describe the quotient ring $\mathbb{Z}[i]/(3)$.

2. Let $G = \text{GL}_n(F_q)$ be the group of invertible $n \times n$-matrices with entries in $F_q$.
   (a) Let $n = 2$, and consider the natural action of $G$ on the set $F_q^2 = F_q \times F_q$ defined by:
   \[
   \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.
   
   Find the stabilizer of the point $(1, 0) \in F_q^2$.
   (b) Find the order of $\text{GL}_2(F_q)$.
   (c) Find the order of $\text{GL}_n(F_q)$. 