Math 320, Fall 2007
Final, December 17

Name: SID:

Instructions
• The total time is 2 hours and 30 minutes.
• The total score is 100 points.
• Use the reverse side of each page if you need extra space.
• Show all your work. A correct answer without intermediate steps will receive no credit.
• Calculators and cheat sheets are not allowed.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
2. Recall that a function \( f : (M, d) \rightarrow (M, d) \) is called a \textit{strict contraction} if there exists a constant \( \alpha < 1 \) such that
\[
d(f(x), f(y)) \leq \alpha d(x, y)
\]
for all \( x, y \in M \).

(a) Is a strict contraction continuous? Uniformly continuous?

(3 points)
(b) Show that for every $x \in M$, the sequence of functional iterates
$\{f^{(n)}(x) : n \in \mathbb{N}\}$ is a Cauchy sequence.

(5 points)
(c) Use parts (a) and (b) to show that if $M$ is complete and $f$ is a strict contraction, then $f$ has a fixed point, i.e., there exists $x_0 \in M$ such that $f(x_0) = x_0$. 

(5 points)

(d) Is the fixed point in part (c) unique? 

(2 points)
2. Recall the normed vector space

\[ \ell_p = \left\{ \mathbf{x} = (x_1, x_2, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}, \]

with norm \[ \|\mathbf{x}\|_p = \left[ \sum_{n=1}^{\infty} |x_n|^p \right]^{\frac{1}{p}}. \]

Is the set

\[ A = \left\{ \mathbf{x} \in \ell_3 : |x_n|^3 \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \right\} \]

compact in \( \ell^3 \)?

(10 points)
3. (a) Describe the closure of the set

$$E = \left\{ \left( x, \cos\left(\frac{1}{x^2}\right) \right) : 0 < x \leq 1 \right\} \subseteq \mathbb{R}^2.$$  

(5 points)
(b) Is the set $\overline{E}$ path-connected, where $E$ is as in part (a)? Explain. 
(10 points)
(c) Repeat the question in part (b) for the closure of the set
\[ F = \left\{ \left( x, \arctan\left( \frac{1}{x^2} \right) \right) : 0 < x \leq 1 \right\}. \]

(5 points)
4. (a) Recall the Hilbert cube

\[ \mathbb{H}^\infty = \{ \mathbf{x} = (x_1, x_2, \cdots) : |x_n| \leq 1 \text{ for all } n \in \mathbb{N} \} \]

with metric \( d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| \).

Show that \( \mathbb{H}^\infty \) is separable, i.e., there exists a countable subset \( A \subseteq \mathbb{H}^\infty \) such that \( \overline{A} = \mathbb{H}^\infty \).

(10 points)
(b) Give an example of a non-separable metric space, with reasons for your answer.

(5 points)
5. Suppose \( \{a_n : n \in \mathbb{N}\} \) is a sequence of strictly positive numbers such that \( \sum_{n=1}^{\infty} a_n \) diverges. Let \( s_n \) denote the \( n \)-th partial sum of this series, i.e., \( s_n = a_1 + \cdots + a_n \).

(a) Determine the convergence or divergence of \( \sum_{n=1}^{\infty} \frac{a_n}{s_n} \).

[Hint: First show that
\[
\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}.
\]

(7 points)
(b) Repeat the question in part (a) with \( \sum_{n=1}^{\infty} a_n/(1 + a_n \sqrt{n}) \).

(8 points)
6. Give brief answers to the following questions:

(5 × 5 = 25 points)

(a) Any connected metric space with more than one point is uncountable. True or false?
(b) There exists a metric space $M$ such that there is no continuous map from $M$ to any other metric space. True or false?
(c) Given any continuous function $f : [a, b] \rightarrow \mathbb{R}$, there exists $c \in [a, b]$ satisfying

$$\int_{a}^{b} f(t) \, dt = f(c)(b - a).$$

True or false?
(d) State the strong nested set property. Give an example of a metric space that does not have the strong nested set property. Provide reasons for your answer.
(e) Let $S = \{ r_1 + r_2 \pi : r_1, r_2 \in \mathbb{Q} \}$. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is $\mathbb{R} \setminus S$?