I Short answer questions: Each question carries 6 marks, your answers should quote the results being used and show your work.

1) Find two integers congruent to 3 mod 5 and 4 mod 7.

2) For which positive integers \( m \) will we have \( 1000 \equiv 1 \mod m \)?

3) Find the least positive residue of \( 1! + 2! + \cdots + 100! \).

4) Suppose that \( n = 81294358X \). Write down a digit in the slot marked \( X \) so that \( n \) is divisible by a) 11 b) 9 c) 4.

5) Find all solutions of \( 7x \equiv 4 \mod 13 \).

6) Define a Carmichael number. Use the necessary and sufficient condition for a number to be a Carmichael number to show that 561 is a Carmichael number.

7) What is the remainder when \( 5^{16} \) is divided by 23?

II State whether the following are true or false with full justification. Each question carries 4 marks.

1) If a positive integer has exactly 3 positive divisors, then it is necessarily of the form \( p^2 \) where \( p \) is a prime.

2) The order \( ord_{19}(5) \) is 7.

3) The number 25 passes Miller’s test for the base 7.

4) The number \( 2^{39} - 1 \) is divisible by 7.

III Find all positive integers \( n \) such that \( n! \) ends with exactly 74 zeros in decimal notation. 14 marks

IV Define the sum of divisors function \( \sigma(n) \) and number of divisors function \( \tau(n) \). Show that there is no positive integer \( n \) with \( \phi(n) = 14 \).

V Show that if \( a \) and \( b \) are relatively prime integers, then \( a^{\phi(b)} + b^{\phi(a)} \equiv 1 \mod ab \). Find the inverse of What is the multiplicative inverse of \( 5^8 \) modulo 16?