Math 310 - Sec. 201 - 2013 - Prof. Juan Souto
Final exam: 8:30 - 11:00

Notation. Throughout this exam, $V$ is a complex vector spaces of finite dimension endowed with an inner product $⟨·,·⟩$. The vector space of all complex polynomials is denoted by $\mathcal{P}$; the subspace consisting of those polynomials of degree at most $n$ is denoted by $\mathcal{P}_n$.

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<td>Total</td>
<td>150</td>
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Question 1. Mark true or false.

<table>
<thead>
<tr>
<th>Expression</th>
<th>True</th>
<th>False</th>
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<tbody>
<tr>
<td>{P(x) \in \mathcal{P}</td>
<td>P(-1) + P(2) = 0} is a subspace of \mathcal{P}.</td>
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<tr>
<td>{P(x) \in \mathcal{P}</td>
<td>P(0) = 1} is a subspace of \mathcal{P}.</td>
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<tr>
<td>{P(x) \in \mathcal{P}</td>
<td>\int_0^1 P(t)dt = 0} is a subspace of \mathcal{P}.</td>
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<td>If ( V \subset \mathbb{C}^n ) is such that ( v + w \in V ) for all ( v, w \in V ), then ( V ) is a subspace.</td>
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<td>Consider ( \mathbb{C}^n ) as a complex vectorspace; ( \mathbb{R}^n ) is a subspace.</td>
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<td>The union of two subspaces ( U_1, U_2 ) of ( V ) is a subspace if and only if either ( U_1 \subset U_2 ) or ( U_2 \subset U_1 ).</td>
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<td>The intersection of three subspaces of ( V ) is a subspace.</td>
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<td>A vector space has infinite dimension if and only if it contains a subspace of dimension ( n ) for all ( n ).</td>
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<td>If ( W \subset V ) is a subspace with ( \dim(W) = \dim(V) ) then ( W = V ).</td>
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<tr>
<td>If ( W_1, W_2 \subset V ) are subspaces with ( \dim(W_1) = \dim(W_2) ) then ( W_1 = W_2 ).</td>
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<tr>
<td>( T : \mathcal{P} \rightarrow \mathcal{P}, \quad T(a_0 + a_1x + \cdots + a_n x^n) = a_0 + a_1x + a_2x^2 ) is linear.</td>
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<tr>
<td>( T : \mathcal{P} \rightarrow \mathbb{C}^3, \quad T(P(x)) = P(1) + P(2) - P(3) ) is linear.</td>
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<td>( T : \mathcal{P} \rightarrow \mathbb{C}, \quad T(a_0 + a_1x + \cdots + a_n x^n) = a_0^2 ) is linear.</td>
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<tr>
<td>( T : \mathcal{P} \rightarrow \mathbb{C}, \quad T(a_0 + a_1x + \cdots + a_n x^n) = a_0 - a_1 + a_2 - a_3 ) is linear.</td>
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<td>If ( T : V \rightarrow V ) is linear and maps a basis of ( V ) to a basis of ( V ), then ( T ) is invertible.</td>
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<td>The linear map ( T : \mathcal{P}_2 \rightarrow \mathbb{C}^5, \quad T(P(x)) = (P(1), P(2), P(3), P(10)) ) is invertible.</td>
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<td>If ( W_1, W_2 \subset V ) are subspaces with ( \dim(W_1) = \dim(W_2) ) then there is a linear map ( T : V \rightarrow V ) with ( T(W_1) = W_2 ).</td>
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<tr>
<td>Statement</td>
<td>True</td>
<td>False</td>
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<td>---------------------------------------------------------------------------</td>
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<tr>
<td>Every linear map $T : V \to V$ has an eigenvalue.</td>
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<td>$T : V \to V$ is surjective if and only if $\ker(T) = 0$.</td>
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<td>If the image of a linear map $T : P_2 \to P_4$ contains 3 linearly independent polynomials, then $T$ is injective.</td>
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<tr>
<td>There is a surjective linear map $T : P_2 \to P_4$</td>
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<td>For every $d = 0, 1, \ldots, 5$ there is a linear map $T : P_4 \to P_4$ whose kernel has dimension $d$.</td>
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<td>If the kernel of a linear map $T : P_n \to P_{n-2}$ has dimension 7 then $T$ is surjective.</td>
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<tr>
<td>There is an injective linear map $T : P_n \to P_{n-2}$.</td>
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<td>There is a unique matrix associated to every linear map $T : V \to W$.</td>
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<td>$T : V \to V$ is diagonalizable if and only if all eigenvalues of $T$ are distinct.</td>
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<td>If all eigenvalues of $T$ are distinct, then $T$ is diagonalizable.</td>
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<td>There is a basis with respect to which the matrix of $T$ is upper triangular.</td>
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<td>$T$ is injective if and only if 0 is not an eigenvalue of $T$.</td>
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<td>$T$ has an eigenvalue if and only if $T$ is normal.</td>
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<tr>
<td>If $T$ is normal, then $T$ is diagonalizable.</td>
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<td>If $T$ is normal, then there is a ON-basis of $V$ consisting of eigenvectors.</td>
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<td>$\lambda \in \mathbb{C}$ is an eigenvalue of $T$ if and only if $\ker((T - \lambda \text{Id})^5) \neq 0$.</td>
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<td>If $T$ is normal and $v$ is an eigenvector of $T$, then $v$ is also an eigenvector of $T^*$.</td>
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<td>If $T^5 = 0$, then $T = 0$.</td>
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<td>If $T^*$ is diagonalizable, then $T$ is diagonalizable.</td>
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<td>Let $T^<em>$ be the adjoint of $T$. If $T^</em> = 0$, then $T = 0$.</td>
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<td>The matrix of $T^*$ with respect to an arbitrary basis of $V$ is the transpose conjugate of that of $T$.</td>
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Question 2. Let $T : V \rightarrow V$ be a linear map.

(1) Define the kernel\(^1\) $\text{Ker}(T)$ of $T$.

(2) Prove that $\text{Ker}(T)$ is a subspace of $V$.

(3) Prove that $T$ is injective if and only if $\text{Ker}(T) = 0$.

\(^1\)or equivalently, the nullspace.
(4) Give an example of a linear map $T : V \to V$ with $\text{Ker}(T) \neq \text{Ker}(T^2) \neq \text{Ker}(T^3)$.

(5) Suppose that $\text{Ker}(T) = \text{Ker}(T^2)$. Prove that $\text{Ker}(T^2) = \text{Ker}(T^3)$. 
Question 3. Given $x_0, \ldots, x_n, y_0, \ldots, y_n \in \mathbb{C}$ suppose that $x_i \neq x_j$ for $i \neq j$. Prove that there is a unique polynomial $P(x) \in \mathcal{P}_n$ of degree at most $n$ satisfying $P(x_i) = y_i$ for all $i = 0, \ldots, n$. 
Problem 4.
(1) Let $v_1, \ldots, v_r \in V$. Define $(v_1, \ldots, v_r)$ is linearly independent.

(2) Let $T : V \to V$ be linear. Suppose that $v_1 \in \ker(T^2)$, $v_2 \in \ker(T - \text{Id})$ and $v_3 \in \ker(T + \text{Id})$ are non-zero vectors. Prove that $(v_1, v_2, v_3)$ is linearly independent.
Question 5. Let $T : V \to V$ be linear and $(v_1, \ldots, v_d)$ a basis of $V$. Prove that the following statements are equivalent:

1. The matrix of $T$ with respect to the basis $(v_1, \ldots, v_d)$ is upper triangular.
2. $T(v_j) \in \text{Span}(v_1, \ldots, v_j)$ for all $j = 1, \ldots, d$. 
Question 6. Let $T : V \to V$ be linear.

(1) Suppose that $T : V \to V$ is diagonalizable. Prove that there is $S : V \to V$ linear with $S^\dim(V) = T$.

(2) Give an example of a complex vector space $V$ of finite dimension and of a non-zero operator $T : V \to V$ with $T^\dim(V) = 0$. 
(3) Suppose that $T : V \to V$ is a non-zero operator with $T^{\dim(V)} = 0$. Prove that there is no operator $S : V \to V$ with $S^{\dim(V)} = T$. 
Question 7. Let $T : V \to V$ be linear.

(1) Define $T$ is normal.

Suppose from now on that $T : V \to V$ is normal and recall that this implies that $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.

(2) Prove that $v \in V$ is an eigenvector of $T$ if and only if it is an eigenvector of $T^*$. 
(3) Suppose that \( v \in V \) is an eigenvector of \( T \). Prove that the orthogonal complement of \( \text{Span}(v) \) is \( T \)-invariant.
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