THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations - December 2011
MATHEMATICS 300

TIME: 3 hours

NO AIDS ARE PERMITTED.
All seven questions are of equal value. Each question is worth 10 points. A passing mark is 28/70. If you obtain a mark of N/70 it will be treated as a mark of N/60 with a maximum score possible of 60/60.

Value

1. (a) Sketch \( S = \{ z \in \mathbb{C} : |z + \frac{1}{2}| > \frac{1}{2} \text{ and } |z + 1| < 1 \} \).

   (b) Find and sketch the image of \( S \) under the mapping \( w = f(z) = 1 + \frac{1}{1 + z} \).

2. (a) Carefully state the Cauchy-Riemann equations and explain their connection with the differentiability and analyticity of a function \( f(z) \).

   (b) Find all entire functions \( f = u + iv \) for which \( v = u^2 \).

3. Let \( f(z) = \frac{z^2}{z^2 + 4z + 3} \).

   (a) Find and classify all singular points of \( f(z) \).

   (b) Determine the residues of \( f(z) \) at each of its singular points.

   (c) Evaluate \( \oint_C f(z) \, dz \) where \( C \) is the positively oriented square with corners at \( \pm i \) and \( -2 \pm i \).

   (d) Where does the Laurent series of \( f(z) \) about \( z = -1 \) converge if it is

       (i) valid near \( z = -1 \)?

       (ii) valid for large \( |z| \)?

   (e) Find the first four nonzero terms of the Laurent series of \( f(z) \) about \( z = -1 \)

       which is valid near \( z = -1 \).

4. (a) Show that the series \( \sum_{n=0}^{\infty} e^{-nz} \) converges uniformly in any half plane \( \text{Re} \, z \geq \delta \),

       for any fixed \( \delta > 0 \).

   (b) Evaluate \( f(z) = \sum_{n=0}^{\infty} e^{-nz} \) for \( \text{Re} \, z > 0 \).

   (c) Evaluate \( \sum_{n=1}^{\infty} n^2 e^{-n} \). Be sure to justify your steps.
5. Consider the improper integral \( I(m) = \int_{0}^{\infty} \frac{x^m}{1+x^8} \, dx \), where the constant \( m \) is an integer, \( m = 0, \pm 1, \pm 2, \ldots \).

(a) For which values of \( m \) does \( I(m) \) converge?
(b) Use contour integration to evaluate \( I(m) \), justifying your calculations.

6. Suppose \( \gamma_1 \) and \( \gamma_2 \) are arcs in the \( z \)-plane that intersect at \( z_0 \). Let \( \alpha \) be the angle of intersection. Let \( w = f(z) \). In the \( w \)-plane, find the angle of intersection at \( w_0 = f(z_0) \) of the image arcs \( f(\gamma_1) \) and \( f(\gamma_2) \) when

(a) \( f(z) = z \);
(b) \( f(z) = 1 + z \);
(c) \( f(z) = 1 + z^3 \);
(d) \( f(z) = 1 + \bar{z} \);
(e) \( f(z) = 1 + \bar{z}^3 \).

[Note that the angle of intersection in the \( w \)-plane could depend on the value of \( z_0 \).]

7. (a) Suppose \( \operatorname{Res}_{z=0} f(z) = A \). Let \( \alpha \) be a complex constant. Evaluate \( \operatorname{Res}_{z=0} f(\alpha z) \).

(b) Evaluate \( \oint_{|z|=2} \frac{z^m}{1+z^3} \, dz \) where the constant \( m \) is an integer, \( m = 0, \pm 1, \pm 2, \ldots \).