MATHEMATICS 300 FINAL EXAM

APRIL 27, 2006. INSTRUCTORS: D. SJERVE, Z. REICHSTEIN

This is a closed book exam. You can use one 8.5” × 11” note sheet but no books or
calculators are allowed. In order to receive credit for a problem you need to show enough
work to justify your answer.

Name (Please print):

Student number:

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Problem 1: Find all complex solutions to the equation \( \cos(z) = 2i \sin(z) \). Express each solution in the form \( z = x + yi \), where \( x \) and \( y \) are real numbers.
(6 marks) **Problem 2:** Answer true or false to the following statements. Give valid reasons for all your answers.

(a) $\log(z^2) = 2 \log(z)$ for every complex number $z$. Here $\log(z)$ denotes the principal value of $\log(z)$.

(b) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$, where $u(x, y)$ and $v(x, y)$ are real valued functions, then the function $w(x, y) = u(x, y) + v(x, y)$ is harmonic.

(c) Suppose $f(z)$ is an analytic function at $z = z_0$ and $f(z_0) = 0$. If $f(z)$ is not identically zero in any disc centered at $z_0$ then $g(z) = \frac{f'(z)}{f(z)}$ has a simple pole at $z_0$. 

(6 marks) Problem 3: Find all singularities of the function \( f(z) = \frac{z}{1 - \cos(z^2)} \). Determine the nature of each singularity (i.e., whether it is removable, essential or a pole). For each pole, determine its order.
(7 marks) **Problem 4:** Prove that the function $f(z) = \bar{z}^{1000}$ is not analytic in any open disc. Here, as usual, $\bar{z}$ denotes the complex conjugate of $z$. 


(7 marks) **Problem 5:** Suppose $f(z)$ is an entire function such that $|f(z)| < 2|z| + 3$ for all complex numbers $z$. Show that $f(z)$ is a polynomial of degree $\leq 1$. 
(6 marks) Problem 6: Suppose the Laurent series for the function \( f(z) = \frac{z}{(z-1)(z-2)} \) in the annulus \( 1 < |z| < 2 \) is given by \( \sum_{j=-\infty}^{\infty} c_j z^j \). Find (a) \( c_{100} \), (b) \( c_{-100} \).
(6 marks) **Problem 7:** Evaluate \[ \int_{0}^{2\pi} \frac{d\theta}{5 - 3\cos(\theta)}. \]
(6 marks) **Problem 8:** Evaluate \( \int_0^\infty \frac{dx}{(x^2 + 1)^3} \).