The University of British Columbia  
Final Examination - December 7, 2005

Mathematics 265  
All Sections

Closed book examination  
Time: 2.5 hours

Special Instructions:

- Be sure that this examination booklet has 4 pages.
- No calculators or notes other than your one page formula sheet are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

• Each candidate should be prepared to produce her/his library/AMS card upon request.
• No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
• Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
• CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
1. [10] Consider the ODE $y' = y(2 - y)$.

   (a) Draw a direction field for this ODE.

   (b) Without solving the equation, determine $\lim_{t \to \infty} y(t)$ if $y(t)$ is a solution of this ODE with

      i. $y(0) = 0$,
      ii. $y(0) = 2$,
      iii. $y(0) = -1$.

2. [10] Consider the ODE

   $$(t - 1) \frac{dy}{dt} = \frac{2t}{t + 1} + 1,$$

   Find the solution of this differential equation that satisfies $y(0) = 0$. Determine the largest interval in which your solution is valid.

3. [20] Consider the initial value problem

   $$y'' + 4y = g(t) \quad y(0) = 3, \quad y'(0) = 1.$$

   (a) Solve the corresponding homogeneous equation, and prove that the two solutions you have found are a fundamental set of solutions.

   (b) For each of the following cases, what form would you guess for a particular solution $Y(t)$, if you were going to use the method of undetermined coefficients? Do not solve the equation!

      i. $g(t) = t^2 e^t$
      ii. $g(t) = e^t \sin t + t$
      iii. $g(t) = \cos(2t)$

   (c) Solve the initial value problem when $g(t) = 5 \cos(t)$. 
4. [15] Let \( f(t) = \sin(t) \) and \( g(t) = e^{-2t} \).
   
   (a) Compute \( W(f, g)(t) \).
   
   (b) Are \( f \) and \( g \) linearly independent on \((0, 2\pi)\)? Justify your answer.
   
   (c) Can one find an ODE \( y'' + p(t)y' + q(t)y = 0 \) where \( p \) and \( q \) are continuous on \((0, 2\pi)\) for which both \( f \) and \( g \) are solutions? Explain.
   
   (d) Compute \( h(t) = f \ast g \) where \( \ast \) denotes convolution.

5. [15] Solve the initial value problem

\[
y'' + 3y' + 2y = (t - 2)u_2(t), \quad y(0) = y'(0) = 0.
\]

What is \( \lim_{t \to \infty} y(t) \)?

6. [15] Consider the system of first-order ODEs given by

\[
\begin{align*}
x'_1 &= -3x_1 + 4x_2 + 4e^{-t} \\
x'_2 &= -x_1 + 2x_2 + 3e^{-t}
\end{align*}
\]

(a) Write this system as a single matrix equation \( \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) \).

(b) Find the general solution of the homogeneous equation \( \mathbf{x}' = A\mathbf{x} \).

(c) Sketch the trajectory on the \( x_1 \times x_2 \)-plane passing through the point \((x_1, x_2) = (4, 2)\).

Specify a point on the \( x_1 \times x_2 \)-plane so that the trajectory passing through this point approaches 0 as \( t \to \infty \).

(d) Now, solve the non-homogeneous system \( \mathbf{x}' = A\mathbf{x} + \mathbf{g}(t) \) with initial conditions \( x_1(0) = 2 \) and \( x_2(0) = 4 \).

7. [15] Solve the initial value problem

\[
\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.
\]
Laplace Transform Table

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2 + a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
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