Math 256. Final

Name:

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE \( y' - y^2 e^{-x} = y^2 \) with \( y(0) = 1 \) has the solution,
   
   \[ (a) \ (e^x + x + C)^{-1} \quad (b) \ (e^{-x} - x + C)^{-1} \quad (c) \ (e^{-x} - x)^{-1} \quad (d) \ (e^{-x} + x)^{-1} \quad (e) \ None \ of \ the \ above, \]
   where \( C \) is an arbitrary constant.

2. The system
   
   \[ y' = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} y \]
   
   has the general solution,
   
   \[ (a) \ u_1 + u_2 e^{2t} + u_3 e^t \quad (b) \ u_1 e^{7t} + u_2 e^t + u_3 e^{-2t} \quad (c) \ u_1 + u_2 e^{2t} + u_3 e^{2t} \quad (d) \ u_1 e^{-t} + u_2 e^{4t} + u_3 e^{2t} \quad (e) \ None \ of \ the \ above, \]
   for three constant vectors \( u_1, u_2 \) and \( u_3 \).

3. The inverse Laplace transform of
   
   \[ \frac{4}{s(s^2 + 4)} \]
   
   \[ (a) \ y(t) = t - \cos 2t, \quad (b) \ y(t) = t + \sin 2t, \quad (c) \ y(t) = 1 - \cos 2t, \quad (d) \ y(t) = 1 - \sin 2t, \quad (e) \ None \ of \ the \ above. \]

4. Which of the following is a solution to the PDE \( u_{tt} = c^2 u_{xx} \):
   
   \[ (a) \ u = \cos cx \sin t \quad (b) \ u = \cos x \sin t \quad (c) \ u = \cos 3x \sin 3ct \quad (d) \ u = e^x \sin ct \quad (e) \ None \ of \ the \ above. \]

5. Laplace transforms are
   
   \[ (a) \ Cool \ if \ you \ like \ that \ sort \ of \ thing \]
   
   \[ (b) \ difficult \ to \ invert \ using \ formal \ definitions \]
   
   \[ (c) \ a \ type \ of \ integral \]
   
   \[ (d) \ a \ method \ of \ turning \ ODEs \ into \ algebraic \ equations \]
   
   \[ (e) \ All \ of \ the \ above. \]
Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. (12 points) A particle in a fusion reactor satisfies the equations of motion,

\[ x' + x + 2y = 0, \quad y' + y - 2x = 0, \quad z' + z = x^2 + y^2. \]

Find the path taken by the particle if it starts at the point \((x(0), y(0), z(0)) = (0, 1, 0)\). Where does the particle eventually end up?

2. (12 points) Write the ODEs

\[ x'' = x + 4y, \quad y'' = 2x + 8y, \]

as a \(2 \times 2\) system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Find the solution if the initial values are \(x(0) = x'(0) = y(0) = 0\) and \(y'(0) = 9\).

3. (12 points) From the definition of the Laplace transform, prove that \(\mathcal{L}\{y''\} = s^2\tilde{y}(s) - sy(0) - y'(0)\) and \(\mathcal{L}\{f(t - a)H(t - a)\} = e^{-as}\tilde{f}(s)\). Find

\[ \mathcal{L}^{-1}\left\{ \frac{s + 4}{(s^2 + 4s + 13)} \right\} \]

Using Laplace transforms, solve the ODE

\[ \ddot{y} + 4\dot{y} + 13y = t^2\delta(t - 2), \]

with \(y(0) = 1\) and \(\dot{y}(0) = 0\), where \(\delta(t)\) is the delta-function.

4. (16 points) Solve

\[ u_t = u_{xx}, \quad u(0, t) = 0, \quad u(x, 0) = \sin \pi x, \]

with

\[ (a) \quad u(2, t) = 0 \quad \& \quad (b) \quad u(2, t) = 2. \]
Fourier Series:

For a periodic function \( f(x) \) with period \( 2L \), the Fourier series is

\[
f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]
\]

with

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.
\]

Helpful trig identities:

\[
sin 0 = \sin \pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),
\]
\[
\cos 0 = -\cos \pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,
\]
\[
\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,
\]
\[
\sin(2A) = 2\sin A \cos A, \quad \sin(A + B) = \sin A \cos B + \cos A \sin B,
\]
\[
\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A + B) = \cos A \cos B - \sin A \sin B,
\]

Useful Laplace Transforms:

\[
f(t) \quad \rightarrow \quad \hat{f}(s)
\]
\[
1 \quad \rightarrow \quad \frac{1}{s}
\]
\[
t^n, \quad n = 0, 1, 2, \ldots \quad \rightarrow \quad n!/s^{n+1}
\]
\[
e^{at} \quad \rightarrow \quad \frac{1}{(s - a)}
\]
\[
\sin at \quad \rightarrow \quad \frac{a}{s^2 + a^2}
\]
\[
\cos at \quad \rightarrow \quad \frac{s}{s^2 + a^2}
\]
\[
t \sin at \quad \rightarrow \quad \frac{2as}{(s^2 + a^2)^2}
\]
\[
t \cos at \quad \rightarrow \quad \frac{(s^2 - a^2)}{(s^2 + a^2)^2}
\]
\[
y'(t) \quad \rightarrow \quad s\tilde{y}(s) - y(0)
\]
\[
y''(t) \quad \rightarrow \quad s^2\tilde{y}(s) - y'(0) - sy(0)
\]
\[
e^{at} f(t) \quad \rightarrow \quad \hat{f}(s - a)
\]
\[
f(t - a)H(t - a) \quad \rightarrow \quad e^{-as}\hat{f}(s)
\]