Math 256 Section 201 Final Exam
Spring 2007
Instructor: Paul A.C. Chang

Last Name: ____________________________________________

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Student Number: _______________________________________

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INSTRUCTIONS:

- Do not lift the cover page until instructed!
- Write your last name, first name, student number, and email address in the spaces above.
- No calculators allowed.
- This exam consists of 10 questions on 16 pages (including this one).
- The maximum score on this exam is 100.
- You have 180 minutes to complete this exam.
- Good Luck!
1) Answer the following questions. You need not show work for this section.

A) What is \(-2 + 3\)? (1 mark)

B) Spencer, Alana, and Jacob equally share 1242 gumballs. How many does each kid get? (1 mark)

C) True or False: The equation \(dx + dy = 0\) is exact. (1 mark)

D) True or False: \(x_0 = 0\) is an ordinary point of the ODE \(x^2y'' + y = 0\). (1 mark)

E) True or False: Metals have low thermal conductivity. (1 mark)

F) True or False: The Special Fundamental Matrix \(\Psi\) satisfies \(\Psi(t + s) = \Psi(t)\Psi(s)\). (1 mark)

G) True or False: The function \(f(x) = \sinh x\) is odd. (1 mark)

H) True or False: Fick's Law describes diffusion. (1 mark)

I) True or False: Laplace's Equation in Cartesian coordinates is \(u_{xx} + u_{yy} + u_{zz} = 0\). (1 mark)

J) True or False: Fourier's Law of Heat Conduction describes the spontaneous transfer of thermal energy through matter, from regions of higher temperature to lower temperature. (1 mark)
2) Let $A$ be an $n \times n$ matrix with eigenvalue $r$, and corresponding eigenvector $\xi$ and corresponding generalized eigenvector $\eta$. Show that $\tilde{x} = te^{rt} \xi + e^{rt} \eta$ solves $\tilde{x}' = A\tilde{x}$. (10 marks)
3) Solve \( y'' + 4y = 4t^2 + 5e^t, \quad y(0) = 5.5, \quad y'(0) = 7. \) (10 marks)
4) Find the solution of $u_{tt} = u_{xx}$ subject to the boundary conditions $u(0, t) = u(1, t) = 0$ and the initial conditions $u(x, 0) = -x(x - 1)$, $u_t(x, 0) = 0$. (10 marks)
4 Cont'd)
5) Consider the following story about Romeo and Juliet. Denote

\[ R(t) = \text{Romeo's love/hate for Juliet at time } t, \]
\[ J(t) = \text{Juliet's love/hate for Romeo at time } t. \]

Positive and negative values correspond to love and hate respectively. Their story is described by the pair of ODEs

\[ R' = aR + bJ, \]
\[ J' = bR + aJ, \]

for some constants \( a < 0 \) and \( b > 0 \).

5a) Give a physical interpretation of the ODEs. (4 marks)
5b) Take $a = -2$ and $b = 3$. Plot a phase portrait for the ODEs. Under what initial conditions do Romeo and Juliet both fall in love with each other? (6 marks)
6) Show that $x_0 = 0$ is a regular singular point of the ODE
\[ x^2 y'' + xy' + \left( x^2 - \frac{1}{4} \right) y = 0. \] (2 marks)

6b) Solve $x^2 y'' + xy' + \left( x^2 - \frac{1}{4} \right) y = 0$ near $x_0 = 0$. (8 marks)
6 Cont'd)
7) Consider the equation \( a u_{xx} - bu_t + cu = 0 \), where \( a, b, c \) are constants. By a suitable change of variables, reduce this equation to a heat equation. (10 marks)
8) Show that \( \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \). (10 marks)

Hint: Consider the Fourier series of \( f(x) = x \) on \([-1,1]\).
9) Consider the modified wave equation

\[ u_{tt} + u = u_{xx}, \quad 0 < x < 1, \quad t > 0 \]

with the boundary conditions

\[ u(0, t) = 0, u(1, t) = 0, \quad t > 0 \]

and the initial conditions

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < 1. \]

Solve for \( u = u(x, t) \). (10 marks)
9 Cont'd)
10) Find the steady state temperature distribution \( T \) on a disk of radius \( a \) which satisfies the boundary condition

\[
T_r(a, \theta) = g(\theta) \text{ for } 0 \leq \theta < 2\pi.
\]

Note that this is a Neumann problem and that its solution is determined only up to an arbitrary additive constant. State a necessary condition on \( g(\theta) \) for this problem to be solvable by the method of separation of variables. What does this condition mean physically? (10 marks)
10 Cont'd)