1. [15 marks] Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & 1 \\ 2 & 3 & 4 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 0 & 2 \\ 1 & 3 & 2 & 2 & 2 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 7 \\ 5 \\ 9 \end{bmatrix}$$

There is an invertible matrix $M$ so that

$$MA = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Mb = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

a) [2 marks] What is rank($A$)?
b) [4 marks] Give the vector parametric form for the set of solutions to $A\mathbf{x} = \mathbf{b}$.
c) [6 marks] Give a basis for the row space of $A$. Give a basis for the column space of $A$. Give a basis for the null space of $A$.
d) [2 marks] Let $A'$ be the $4 \times 5$ matrix obtained by deleting the 5th column of $A$ from $A$. What is the rank of $A'$?

2. [15 marks] Let

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Determine an orthonormal basis of eigenvectors and hence an orthogonal matrix $Q$ and a diagonal matrix $D$ so that $A = QDQ^T$. You may find it useful to know that 5 is an eigenvalue of $A$.

3. [7 marks] Determine the matrix $A$ corresponding to the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ of projection onto the vector $(1, 2, 3)^T$. 


4. [8 marks] Consider the $2 \times 2$ matrix $A$ as follows

$$A = \begin{bmatrix} -4 & 4 \\ -12 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$ 

Define $a_n, b_n, c_n, d_n$ using

$$A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$$

Compute

$$\lim_{n \to \infty} \frac{a_n}{b_n}, \quad \lim_{n \to \infty} \frac{c_n}{d_n}.$$ 

5. [10 marks] You are attempting to solve for $x, y, z$ in the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$ 

Find a ‘least squares’ choice $\hat{b}$ in the column space of $A$ (and hence with $||b - \hat{b}||^2$ being minimized) and then solve the new system $Ax = \hat{b}$ for $x, y, z$. 

6. [15 marks] The differentiation operator $\frac{d}{dx}$ maps (differentiable) functions into functions. The operator can be viewed as a linear transformation on the vector space of differentiable functions. Consider the 3-dimensional vector space $P_2$ of all polynomials in $x$ of degree at most 2. Then two possible bases for $P_2$ are $V = \{1, x, x^2\}$ and $U = \{1 + x, x + x^2, x^2 + 1\}$.

a) [5 marks] Give the $3 \times 3$ matrix $A$ representing the linear transformation $\frac{d}{dx}$ acting on $P_2$ with respect to the basis $V$.

b) [5 marks] Give the matrix $B$ representing $\frac{d}{dx}$ with respect to the basis $U$. You may find it helpful to note that

$$1 = \frac{1}{2}(1 + x) - \frac{1}{2}(x + x^2) + \frac{1}{2}(x^2 + 1)$$

$$x = \frac{1}{2}(1 + x) + \frac{1}{2}(x + x^2) - \frac{1}{2}(x^2 + 1)$$

$$x^2 = -\frac{1}{2}(1 + x) + \frac{1}{2}(x + x^2) + \frac{1}{2}(x^2 + 1)$$

c) [5 marks] Is matrix $A$ diagonalizable?
7. [10 marks] Let $V$ be a finite dimensional vector space and assume $X = \{x_1, x_2, \ldots, x_k\}$ is a linearly independent set of $k$ vectors and assume $Y = \{y_1, y_2, \ldots, y_k, y_{k+1}\}$ is a linearly independent set of $k+1$ vectors. Then show that there is some vector in $Y$, say $y_j$, so that $\{x_1, x_2, \ldots, x_k, y_j\}$ is a linearly independent set of $k+1$ vectors.

8. [10 marks] For what values of $k$ is the following matrix diagonalizable?

$$A = \begin{bmatrix} 2 & 0 & 0 \\ k & 1 & 1 \\ 2 & -2 & 4 \end{bmatrix}$$

Hint: determine eigenvalues for $A$. What is required to make $A$ diagonalizable?

9. [10 marks]

a) [4 marks] Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for $\mathbb{R}^3$. For any $v \in \mathbb{R}^3$, if $v = c_1 v_1 + c_2 v_2 + c_3 v_3$ then show that $v^T v = ||v||^2 = c_1^2 + c_2^2 + c_3^2$.

b) [6 marks] Let $A$ be a symmetric $3 \times 3$ matrix with eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$. Show that

$$\lambda_1 = \max_x x^T A x$$

where the maximum is taken over all vectors $x \in \mathbb{R}^3$ with $x^T x = 1$.

100 Total marks