Math 221: Final Exam
Date: December 6, 2008

Name (please print):

Student Number:

Section number:

Instructions: No textbook, notes, or other aids allowed. Show all your work. If you need more space, use the back of the page. Each problem is worth 10 marks (5+5).

There are 14 pages in this exam. The last 3 pages are blank for scratch work. Please return all 14 pages.
Problem 1a: [5] Let \( A \) denote the matrix \( A = \begin{pmatrix} 3 & 0 & -2 \\ 1 & -2 & 3 \\ 4 & -2 & 1 \end{pmatrix} \). Find a basis for the column space of \( A \).

Problem 1b: [5] Determine whether the vector \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) is in the column space of the matrix \( A \) above.
Problem 2a: [5] Let \( A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & -2 & h \end{pmatrix} \). Find all values of \( h \) such that \( A \) is NOT invertible.

Problem 2b: [5] True or false (explain your answer): if \( A \) is an \( n \times n \) matrix such that \( \lambda = 0 \) is an eigenvalue of \( A \), then \( A \) is not invertible.
Problem 3: [5] Show that the determinant of the matrix 
\[ A = \begin{pmatrix} 
1 & x & x^2 \\
1 & y & y^2 \\
1 & z & z^2 
\end{pmatrix} \]
is given by \((z - x)(y - x)(z - y)\).

Problem 3b: [5] Find the determinant of the matrix 
\[ A = \begin{pmatrix} 
3 & 3 & 3 \\
1 & 5 & 25 \\
10 & 70 & 490 
\end{pmatrix}, \]
(Hint: you can use part (a) above to save some calculation.)
Problem 4a: [5] Let $W$ be the plane in $\mathbb{R}^4$ spanned by the vectors $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

and $w = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$. Verify that the vector $u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ is in the subspace $W$ and show that the vectors $u$ and $w$ form an orthogonal basis for $W$.

Problem 4b: [5] Find the vector in $W$ which is closest to $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$
Problem 5a: [5] Find the eigenvalues and eigenvectors for \( A = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} \).

Problem 5b: [5] Find the matrix \( A^{1000} \). (Simplify your answer as much as possible.)
Problem 6a: [5] Suppose that each year 10% people living in Alberta move to BC, while 15% of people living in BC move to Alberta. Write a transition matrix that represents the change in population each year.

Problem 6b: [5] Find limiting population distribution if in the initial year, there at 100,000 people living in each province BC and Alberta.
Problem 7a: [5] Find the characteristic polynomial of the matrix \( \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{pmatrix} \).

Give your answer in the form \(-\lambda^3 + a\lambda^2 + b\lambda + c\) where \(a, b, c\) are real numbers.

Problem 7b: [5] True or false (explain your answer): If \(A\) and \(B\) are similar matrices, then \(A\) and \(B\) have the same eigenvalues.
Problem 8a: [5] Suppose $A$ and $B$ are square matrices with $AB = BA$, and $B$ invertible. Then if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, show that $Bv$ is an eigenvector of $A$ with the same eigenvalue.

Problem 8b: [5] True or false (explain your answer): let $T$ be the transformation of $\mathbf{R}^2 \to \mathbf{R}^2$ given by $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$. Then $T$ is a linear transformation.
Problem 9a: [5] Let $B = \{b_1, b_2\}$ be a basis for $\mathbb{R}^2$ and let $T$ be the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ such that $T(b_1) = 2b_1 + b_2$ and $T(b_2) = b_2$. Find the matrix of $T$ relative to the basis $B$.

Problem 9b: [5] Suppose now that $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find the matrix of $T$ relative to the standard basis of $\mathbb{R}^2$. 

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Problem 10a: True or false (explain your answer): Suppose that $v_1, \ldots, v_n$ are a basis for $\mathbb{R}^n$, and $A$ is an invertible $n \times n$ matrix. Then the vectors $Av_1, \ldots, Av_n$ are also a basis for $\mathbb{R}^n$.

Problem 10b: True or false (explain your answer): if $A$ is a $2 \times 2$ matrix with characteristic polynomial $(\lambda - 2)^2$ then it is diagonalizable.