Marks

1. [Short answer]

(a) [3] Let \( A = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \) and \( B = \begin{bmatrix} s & -t \\ t & s \end{bmatrix} \). Is it true that \( AB = BA \) for all choices of \( x, y, s \) and \( t \)?

(b) [3] Write down a \( 2 \times 2 \) matrix with real entries but with complex eigenvalues.

(c) [3] For which values of \( a \) does \( \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \) have only one eigenvector (up to scalar multiples)?

(d) [3] If \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) what is \( A^{100} \)?

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2. For which values of $a$ and $b$ are the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$ linearly independent?
3. Let $T$ be the triangle in three dimensional space with vertices located at $p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $q = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $r = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(a) What is the cosine of the angle at the vertex $p$?

(b) What is the area of the triangle. (Hint: it is half the area of the parallelogram spanned by two of its sides.)
4. Consider the quadratic function

\[ f(x, y) = 2x^2 + y^2 + 2xy - 8x - 6y + 16 \]

Find the minimum value of \( f \) and where it occurs.
5. Let $T$ be the linear transformation from three dimensional space $\mathbb{R}^3$ to $\mathbb{R}^3$ with

\[
T \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

What is the matrix for $T$?
6. Find the determinant of each of the following matrices and decide whether they are invertible.

(a) \[
\begin{bmatrix}
1 & 2 & 1 & -1 & 4 & 9 \\
0 & 2 & 0 & 0 & 8 & 8 \\
0 & 0 & 1 & 1 & 5 & -5 \\
0 & 0 & 0 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 2 & 1
\end{bmatrix}
\]

(b) The matrix product

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
6 & 2
\end{bmatrix}
\]

(c) Any 3 \times 3 matrix with eigenvalues 1, 2 and 3. Give a reason.
The matrix $P$ given by

$$P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{3} & 0 \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{2}
\end{bmatrix}$$

contains the transition probabilities for a random walk on three sites. The eigenvalues of $P$ are $1$, $1/3$ and $0$.

(a) \[8\] Find the eigenvector of $P$ corresponding to the eigenvalue $1$.

(b) \[8\] If the initial probabilities are given by a vector $x$ with positive entries that sum to $1$, find the limiting probabilities $\lim_{n \to \infty} P^n x$. 

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8. Solve the system of differential equations

\[ \begin{align*}
    x'_1(t) &= -x_1(t) + 2x_2(t) \\
    x'_2(t) &= -2x_1(t) - x_2(t)
\end{align*} \]

with initial conditions \( x_1(0) = 1, x_2(0) = 1 \). Write your final answer in a form that does not involve complex numbers.
Special Instructions:

No calculators, cell phones, or books are allowed.
You may bring one letter-sized formula sheet.
For all questions except the first, you must show your work (i.e.,
intermediate steps) for full credit.

Rules governing examinations

1. Each candidate should be prepared to produce his or her library/AMS card upon request.
2. Read and observe the following rules:
   No candidate shall be permitted to enter the examination room after the expiration of one half
   hour, or to leave during the first half hour of the examination.
   Candidates are not permitted to ask questions of the invigilators, except in cases of supposed
   errors or ambiguities in examination questions.
   CAUTION - Candidates guilty of any of the following or similar practices shall be immediately
   dismissed from the examination and shall be liable to disciplinary action.
   (a) Making use of any books, papers or memoranda, other than those authorized by the
   examiners.
   (b) Speaking or communicating with other candidates.
   (c) Purposely exposing written papers to the view of other candidates. The plea of accident or
   forgetfulness shall not be received.
3. Smoking is not permitted during examinations.

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