Math 121 Formula Sheet

1. Area as limit of Riemann sums:
   The area $R$ lying under the graph $y = f(x)$ of a non-negative continuous function $f$ between the vertical lines $x = a$ and $x = b$ is given by
   
   $$ R = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i, $$

   where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n < b$ is a partition of $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$.  

2. Trapezoid Rule
   The $n$-subinterval trapezoid rule approximation to $\int_a^b f(x) \, dx$, denoted $T_n$, is given by
   
   $$ T_n = h \left( \frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2} \right), \text{ where } y_i = f(x_i). $$

   If $f$ has a continuous second derivative on the interval $[a, b]$ satisfying $|f''(x)| \leq K$ there, then the error in applying trapezoid rule is at most $K(b-a)^3/(12n^2)$.

3. Midpoint Rule
   If $h = (b-a)/n$, let $m_j = a + (j - \frac{1}{2})h$ for $i \leq j \leq n$. The midpoint rule approximation to $\int_a^b f(x) \, dx$, denoted $M_n$, is given by
   
   $$ M_n = h \sum_{j=1}^{n} f(m_j). $$

   If $f$ has a continuous second derivative on the interval $[a, b]$ satisfying $|f''(x)| \leq K$ there, then the error in applying midpoint rule is at most $K(b-a)^3/(24n^2)$.

4. Simpson’s Rule
   The Simpson’s rule approximation to $\int_a^b f(x) \, dx$ based on a subdivision of $[a, b]$ into an even number $n$ of subintervals of equal length $h = (b-a)/n$ is denotes $S_n$ and is given by
   
   $$ S_n = \frac{h}{3} \left( y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right). $$

   If $f$ has a continuous fourth derivative on the interval $[a, b]$ satisfying $|f^{(4)}(x)| \leq K$ there, then the error in applying Simpson’s rule is at most $K(b-a)^5/(180n^4)$.

5. Pappus’s Theorem
   (a) If a plane region $R$ lies on one side of a line $L$ in that plane and is rotated about $L$ to generate a solid of revolution, then the volume $V$ of that solid is given by
   
   $$ V = 2\pi \bar{r} A, $$

   where $A$ is the area of $R$ and $\bar{r}$ is the perpendicular distance from the centroid of $R$ to $L$.  

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(b) If a plane curve $C$ lies on one side of a line $L$ in that plane and is rotated about that line to generate a surface of revolution, then the area $S$ of that surface is given by

$$S = 2\pi \bar{r}s,$$

where $s$ is the length of the curve $C$, $\bar{r}$ is the perpendicular distance from the centroid of $C$ to the line $L$.

6. The general normal distribution

A random variable $X$ on $(-\infty, \infty)$ is said to be normally distributed with mean $\mu$ and standard deviation $\sigma$ (where $\mu$ is any real number and $\sigma > 0$) if its probability density function $f_{\mu,\sigma}$ is given by

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

7. Taylor polynomials and remainder

If the $(n + 1)$st derivative of $f$ exists on an interval containing $c$ and $x$ and if

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x - c)^k$$

is the Taylor polynomial of degree $n$ for $f$ about $x = c$, then

$$f(x) = P_n(x) + E_n(x),$$

where

$$E_n(x) = \frac{1}{n!} \int_{c}^{x} (x - t)^n f^{(n+1)}(t) \, dt.$$

8. Fourier series

If $f(t)$ is a period function with fundamental period $T$, is continuous with a piecewise continuous derivative, then for every $t$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n \omega t) + b_n \sin(n \omega t)), \quad \omega = \frac{2\pi}{T},$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n \omega t) \, dt, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n \omega t) \, dt, \quad n = 1, 2, 3 \ldots.$$