

# Numerically Solving the Semi-Infinite Hydraulic Fracture Problem

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This summer I worked with Professor Anthony Pierce, attempting to numerically solve the semi-infinite hydraulic fracture problem. Hydraulic fracturing is the process by which pressurized fluid is used to propagate fractures in rock. One of its uses is in the oil and gas industry, where it facilitates the removal of hydrocarbons. There are however dangers to hydraulic fracturing. The fracture could break into the water table, allowing the hydrocarbons to seep into the water supply. There are also concerns that hydraulic fracturing has led to a higher incidence of earthquakes. A solid understanding of how these fractures propagate therefore is of utmost importance.

I was studying the propagation of the fracture along one dimension. Furthermore, I assumed that the fracture continued from its tip, out to infinity. Solving this class of problems is crucial to efficiently solving the general three-dimensional fracture problem.

Solving this hydraulic fracture problem amounts to knowing both the pressure of the fluid, and the width of the fracture as a function of distance from the tip of the fracture. After resealing, the equations governing the behaviour of the crack can be summarized as:

$$\pi = \frac{-1}{4\pi} \int_0^\infty \frac{\Omega'(s)}{s-x} dx \quad (1)$$

$$\frac{d\pi}{dx} = \frac{1}{\Omega(x)^2} \quad (2)$$

where  $\pi$  and  $\Omega$  are the resealed pressure and width respectively. Because the width of the fracture goes to zero at the tip, equation 2 makes typical numerical analysis of this problem difficult.

It can be shown through physical and mathematical reasoning that  $\Omega(x) \sim x^{1/2}$  close to the tip of the fracture, and  $\Omega(x) \sim x^{2/3}$  far from the tip. The problem is then to solve the system of equations in the intermediate fracture region, so that the solution matches this behaviour in the near, and far-tip regions.

We first partition the fracture into  $N + 1$  intervals using the mesh-points  $x_1, \dots, x_n$ , where  $x_1$ , and  $x_n$  are chosen so that  $\Omega(x) \sim x^{1/2}$  for  $x \in [0, x_1]$ , and  $\Omega(x) \sim x^{2/3}$ , for  $x \in [x_n, \infty]$ . Assuming that  $\Omega$  is quadratic on each interval, and forcing continuity at the endpoints, we get a system of  $3N - 4$  non-linear equations, in the unknowns  $\Omega_2, \dots, \Omega_{n-1}, \Omega'_2, \dots, \Omega'_{n-1}, \pi_1, \dots, \pi_n$ , where  $\pi_i = \pi(x_i)$ ,  $\Omega_i = \Omega(x_i)$ , and  $\Omega'_i = \Omega'(x_i)$ . We then use Newton's method to solve this system. For the initial condition, we used the data obtained from others using a different algorithm. We then used this data to interpolate the values at our own mesh points.

Unfortunately, the solution we got using this algorithm did not match the expected behaviour. After working with this model for a couple of weeks without success, we decided to test it on a much simpler problem which could be solved analytically. This problem assumed a finite fracture, with a constant fluid pressure. The simplicity of this problem actually linearizes all the equations, so the the system can be solved by just inverting a matrix. The solution obtained from this method again did not converge to the solution obtained analytically. Since the algorithm did not work on the simplified problem we decided to try another method.

The next algorithm we tried assumed that  $\Omega$  was linear on each interval. This algorithm actually converged very quickly to the proper solution on the simplified finite fracture, but we could not get it to work on the semi-infinite fracture. The last algorithm we used was adapting a Hermite-Cubic Collocation Scheme that Professor Pierce had used to solve a different class of fracture problems. The algorithm essentially assumes that  $\Omega$  is cubic on each of the intervals. We performed all the calculations necessary to implement the algorithm, but unfortunately we ran out of time to properly adapt the MATLAB code.

Although I did not get any results this summer, I learned a lot. I very little previous experience with numerical methods. Mathematica was completely new to me, it was a great opportunity to learn the program, and apply to it a sophisticated problem. I switched to MATLAB near the end of the summer, but did not have time to become confident using it. I think the mathematical and programming exposure I got this summer will greatly help me in the future.