Problem 1. Prove that if \( a, b, c \in \mathbb{C} \) and the following relations are satisfied:
- \( a + b + c = 0 \); and
- \( |a| = |b| = |c| \),
then \( a^3 = b^3 = c^3 \).

Can this result be extended to more than 3 complex numbers?

Solution. Clearly, if \( a = 0 \), then \( b = c = 0 \). So, from now on, we assume neither number is 0 and then dividing by \( a \), we may assume we deal with the complex numbers 1, \( r, s \) with \( r = e^{i\alpha} \) and \( s = e^{i\beta} \) for some real numbers \( \alpha, \beta \) such that
\[ 1 + r + s = 0, \]
which means
\[ \sin(\alpha) + \sin(\beta) = 0 \] and so, \( \sin(\beta) = -\sin(\alpha) \), which in particular, yields \( \cos(\beta) = \pm \cos(\alpha) \).

In conclusion, \( r \) and \( s \) are the two primitive third roots of unity, which yields that \( a^3 = b^3 = c^3 \) as desired.

Now, if we deal with more than 3 complex numbers, it will not be enough to assume that \( a_1 + \cdots + a_n = 0 \) and \( |a_1| = |a_2| = \cdots = |a_n| \) in order to conclude that \( a_1^n = a_2^n = \cdots = a_n^n \). Indeed, we can let
\[ a_k = e^{2\pi ik/(n-2)} \text{ for } k = 1, \ldots, n-2 \]
and \( a_n = -a_{n-1} \) for some complex number \( a_{n-1} \) which is not a root of unity.

Problem 2. If the series \( \sum_{n=1}^{\infty} a_n \) of real numbers converges, does \( \sum_{n=1}^{\infty} a_n^3 \) converge?

Solution. No; here’s a counterexample. For each positive integer \( n \), we let \( a_{2n} = \frac{1}{\sqrt{n}} \).

Now, for each \( n \geq 1 \) and for each \( 1 \leq k < 2^n \), we let \( a_{2^n+k} = -\frac{1}{\sqrt{n} \cdot 2^n} \).

For completion, we let \( a_1 = 0 \).

Claim 1. The series \( \sum_{k=1}^{\infty} a_k \) converges.

Proof of Claim 1. For each \( 1 < \ell < m \), we let \( n_1 \) be the unique positive integer such that \( 2^{n_1} \leq \ell < 2^{n_1+1} \) and also, we let \( n_2 \) be the unique positive integer such that \( 2^{n_2} \leq m < 2^{n_2+1} \); then we let \( k_1 := \ell - 2^{n_1} \) and \( k_2 := m - 2^{n_2} \).

We have
\[ \sum_{k=\ell}^{m} a_k \leq \frac{1}{\sqrt{n_1}} + \frac{1}{\sqrt{n_2}} + \sum_{i=n_1+1}^{n_2-1} \frac{1}{2^i \cdot \sqrt{i}}, \]
and so, if \( n_1, n_2 > N \), then
\[ \sum_{k=\ell}^{m} a_k \leq \frac{2}{\sqrt{N}} + \frac{1}{2N} \to 0 \text{ as } N \to \infty. \]
So, indeed, $\sum_{k=1}^{\infty} a_k$ converges.

**Claim 2.** The series $\sum_{k=1}^{\infty} a_k^3$ diverges.

Indeed, for each $n \geq 1$ and for each $0 \leq k \leq 2^n - 1$, we have

$$\sum_{k=1}^{2^n} a_k^3 > \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{2^i}{i \cdot 8^i} > \sum_{i=1}^{n} \frac{1}{2^i},$$

which diverges to $\infty$, thus proving that $\sum_{k=1}^{\infty} a_k$ diverges.

**Problem 3.** For what pairs $(a,b)$ of positive real numbers we have that the integral

$$\int_b^{\infty} \left( \sqrt{x + a} - \sqrt{x} - \sqrt{x} - \sqrt{x - b} \right) \, dx$$

converges.

**Solution.** The key observation is that

$$\sqrt{x + a} - \sqrt{x} = \frac{a}{\sqrt{x + a} + \sqrt{x}} = \frac{a}{2\sqrt{x} \cdot (\sqrt{x} + \sqrt{x + a})},$$

in other words,

$$\left| \sqrt{x + a} - \sqrt{x} - \frac{a}{2\sqrt{x}} \right| < \frac{a^2}{2x^2}.$$

So, $\sqrt{x + a} - \sqrt{x} = \sqrt{\frac{a^2}{2x^2}} + f_a(x)$, where $|f_a(x)| < C_a x^{-5/4}$ for some positive constant $C_a$ depending only on $a$ (and independent of $x$). A similar computation yields that

$$\sqrt{x} - \sqrt{x - b} = \sqrt{\frac{b^2}{2x^2}} + f_b(x),$$

where $|f_b(x)| < C_b x^{-5/4}$ for some positive constant $C_b$ depending only on $b$ (and independent of $x$). This means that

$$\int_b^{\infty} |f_a(x) - f_b(x)| \, dx < \int_b^{\infty} |f_a(x)| \, dx + \int_b^{\infty} |f_b(x)| \, dx < \infty.$$

So, we conclude that $\int_b^{\infty} \left( \sqrt{x + a} - \sqrt{x} - \sqrt{x} - \sqrt{x - b} \right) \, dx$ converges if and only if $\int_b^{\infty} \frac{\sqrt{a - \sqrt{x}}}{\sqrt{2 \sqrt{x}} \sqrt{x}} \, dx$ converges, which happens if and only if $a = b$.

**Problem 4.** For each $n \in \mathbb{N}$, we let $S_n$ be the set of all pairs $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ with the property that $x^3 - 3xy^2 + y^3 = n$.

(a) For each $n \in \mathbb{N}$, prove that either $S_n$ is the empty set, or it has at least 3 elements.

(b) Prove that $S_{2021}$ is the empty set.

**Solution.**
(a) We observe that once \((x, y)\) is a solution, then also \((-y, x - y)\) is a solution and therefore, also \((y - x, -x)\) is a solution; finally, applying the transformation \((x, y) \mapsto (-y, x - y)\) to the last solution \((y - x, -x)\), we recover the original solution \((x, y)\). We note that for a solution \((x, y)\), the other two solutions \((y - x, -x)\) and \((-y, x - y)\) are distinct because otherwise we would have \(y = -x\) and \(x = y - x = -x - x\), i.e., \(x = 0\) and so, \(y = 0\), contradicting the fact that \(x^3 - 3xy^2 + y^3 = n\) cannot have the trivial solution. So, indeed, once there exists a solution, then there are at least 3 solutions.

The idea for this solution comes from looking at transformations of the form \((x, y) \mapsto (ax + by, cx + dy)\) which preserve the quantity \(x^3 - 3xy^2 + y^3\); also, we search for small values for \(a, b, c, d\).

(b) Using Fermat’s Little Theorem, we have \(x^3 \equiv x \pmod{3}\) and so,
\[
2 \equiv 2021 \equiv x^3 - 3xy^2 + y^3 \equiv x + y \pmod{3}
\]
and so, noting part (a) above, we may assume \(x\) is divisible by 3 and therefore, \(y \equiv 2 \pmod{3}\). But then \(y^3 \equiv 8 \pmod{9}\) and overall (because 3 divides \(x\)),
\[
x^3 - 3xy^2 + y^3 \equiv 8 \not\equiv 2021 \pmod{9},
\]
contradiction. Therefore, there are no solutions to \(x^3 - 3xy^2 + y^3 = 2021\).