Problem 1. Let $a$ and $s$ be real numbers satisfying the following properties:

- $0 < a \leq 1$; and
- $s > 0$, but $s \neq 1$.

Prove that $\frac{1 - \frac{s^a}{s}}{1 - \frac{1}{s}} \leq (1 + s)^{a-1}$.

Solution. We observe that replacing $s$ by $1/s$ yields the same inequality since

$$\frac{1 - \frac{s^a}{s}}{1 - \frac{1}{s}} = \frac{1 - s^a}{s^{a-1} \cdot 1 - s}$$

and $(1 + 1/s)^{a-1} = \frac{1}{s^{a-1}} \cdot (1 + s)^{a-1}$. So, from now on, we assume $0 < s < 1$. Also, we may assume $0 < a < 1$ because the case $a = 1$ is clear. We let

$$f_a(s) := -(1 - s^a) + (1 - s) \cdot (1 + s)^{a-1} \text{ for } 0 < s < 1.$$ We observe that $f_a(0) = 0 = f_a(1)$ and so, in order to prove that $f_a(s) > 0$ for $0 < s < 1$, it suffices to prove that there exists a unique $d \in (0, 1)$ such that $f_a$ is increasing on $(0, d)$ and then $f_a$ is decreasing on $(d, 1)$; this will guarantee that $f_a(s) > 0$ for all $0 < s < 1$.

So, we compute

$$f_a(s)' = as^{a-1} - (1 + s)^{a-1} + (1 - s) \cdot (a - 1) \cdot (1 + s)^{a-2}$$

$$= s^{a-1} \cdot (a - (1 + 1/s)^{a-1} - (1 - 1/s) \cdot (a - 1) \cdot (1 + 1/s)^{a-2})$$

$$= s^{a-1} \cdot g_a(1/s),$$

where $g_a(x) := a - (1 + x)^{a-1} - (1 - x) \cdot (a - 1) \cdot (1 + x)^{a-2}$, which is defined for $x > 1$ (note that $x$ corresponds to $1/s$, where $0 < s < 1$). Once again we differentiate:

$$g_a'(x) = -(a - 1) \cdot (1 + x)^{a-2} + (a - 1) \cdot (1 + x)^{a-2} - (1 - x) \cdot (a - 1) \cdot (a - 2) \cdot (1 + x)^{a-3}$$

$$= -(a - 1)(a - 2) \cdot (1 - x) \cdot (1 + x)^{a-3} > 0$$

since $0 < a < 1$ and $x > 1$. On the other hand, $g_a(1) = a - 2^{a-1}$ and we view it as a function of $a$, i.e.,

$$h(a) := a - 2^{a-1} \text{ for } 0 < a < 1.$$ We have

$$h'(a) = 1 - \ln(2) \cdot 2^{a-1},$$

which is decreasing and its smallest value is obtained for $a = 1$ and then

$$h'(a) > h'(1) = 1 - \ln(2) > 0 \text{ for } 0 < a < 1.$$

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Now, we select the disks $D_i$ for $j = 1, \ldots, m$ as follows: $i_j$ is the smallest index $k$ with the property that $D_k$ is not contained in $\bigcup_{c \in C_k} C'_c$. So, $i_1 := 1$ and clearly, $m \leq n$ (i.e., the above process is destined to end in finitely many steps and at one moment there is no additional disk we can select in our process). Hence, 

$$\bigcup_{i=1}^n D_i \subseteq \bigcup_{j=1}^m D'_{i_j}$$

and so, the area of $\bigcup_{j=1}^m D'_{i_j}$ is $\frac{1}{9}$ times the area of $\bigcup_{j=1}^m D'_{i_j}$ and so, the area of $\bigcup_{i=1}^n D_{i_j}$ is at least $\frac{1}{5}$ times the area of $\bigcup_{i=1}^n D_i$. 


Problem 2. Let $S$ be the set of all real numbers of the form $\frac{m+n}{\sqrt{m^2+n^2}}$ where $m$ and $n$ are positive integers. Prove that for each two distinct elements $u < v$ contained in $S$, there exists another element $w \in S$ such that $u < w < v$.

Solution. We observe that letting $r := \frac{m}{n}$ (where $m \leq n$), then

$$\frac{m+n}{\sqrt{m^2+n^2}} = \frac{1+r}{\sqrt{1+r^2}}.$$ 

So, we let $f(r) := \frac{1+r}{\sqrt{1+r^2}}$ for all rational numbers $0 < r \leq 1$. Now, we observe that the above function $f$ is increasing since

$$f'(x) = \frac{1 - x}{(1 + x^2)^{3/2}} > 0$$

if $0 < x < 1$. So, for any distinct elements $u < v$ in $S$, there exist $0 < r_1 < r_2 \leq 1$ such that $u = f(r_1)$ and $v = f(r_2)$. Hence, $w := f \left( \frac{1+r_1}{\sqrt{1+r_1^2}} \right) \in S$ and $u < w < v$.


Problem 3. We consider a set $S$ of finitely many disks in the cartesian plane (of arbitrary centers and arbitrary radii) and we let $A$ be the area of the region represented by their union. Prove that there exists a subset $S_0 \subseteq S$ satisfying the following two properties:

- any two disks from $S_0$ are disjoint.
- the sum of the areas of the disks from $S_0$ is at least $\frac{4}{9}$.

Solution. We order the radii of the given disks $D_1, \ldots, D_n$ in decreasing order $r_1 \geq r_2 \geq \cdots \geq r_n$. Also, we let $D'_i$ (for $i = 1, \ldots, n$) be the disks with the same centers $O_i$ as the corresponding disk $D_i$ but with its radii $r'_i := 3r_i$ for $i = 1, \ldots, n$. Now, we select the disks $D_{i_j}$ for $j = 1, \ldots, m$ as follows: $i_j$ is the smallest index $k$ with the property that $D_k$ is not contained in $\bigcup_{c \in C_k} C'_c$. So, $i_1 := 1$ and clearly, $m \leq n$ (i.e., the above process is destined to end in finitely many steps and at one moment there is no additional disk we can select in our process). Hence, 

$$\bigcup_{i=1}^n D_i \subseteq \bigcup_{j=1}^m D'_{i_j}$$

and so, the area of $\bigcup_{j=1}^m D'_{i_j}$ is $\frac{1}{9}$ times the area of $\bigcup_{j=1}^m D'_{i_j}$ and so, the area of $\bigcup_{i=1}^n D_{i_j}$ is at least $\frac{1}{5}$ times the area of $\bigcup_{i=1}^n D_i$. 


On the other hand, we claim that there are no points in common for the disks $D_1, \ldots, D_m$. Indeed, if there is a point $x$ in common for the disks $D_k$ and $D_\ell$ for $k < \ell$, then we have that for each point $y$ in the disk $D_\ell$,
\[
\text{dist}(O_k, y) 
\leq \text{dist}(O_k, x) + \text{dist}(x, O_\ell) + \text{dist}(O_\ell, y) 
\leq r_k + r_\ell + r_\ell 
\leq r_k + 2r_\ell 
\leq 3r_\ell,
\]
which yields that $y$ is contained in $D'_k$. In other words, $D_k$ is contained in $D'_\ell$, where $k < \ell$; this contradicts our choice of $i_\ell$ which has the property that $D_{i_\ell}$ is not contained in $\bigcup_{j < \ell} D'_{i_j}$. In conclusion, the disks $D_{i_j}$ (for $j = 1, \ldots, m$) are indeed disjoint and the sum of their areas is at least $\frac{1}{4}$ times the area of the union of all disks $D_1, \ldots, D_n$.

**Problem 4.** Let $\{u_n\}_{n \geq 1}$ be a recurrence sequence defined by $u_{n+1} = \frac{\sqrt[4]{64L+15}}{4}$ for each $n \geq 1$. Find $\lim_{n \to \infty} u_n$.

**Solution.** If there exists a limit $L$ to the above sequence, then we must have
\[
L = \frac{\sqrt[4]{64L + 15}}{4},
\]
i.e., $64L^3 = 64L + 15$, which suggests that the sequence either converges to one of the roots of the above equation, or that the sequence diverges to $\pm \infty$. On the other hand, since the function $f(x) := \frac{\sqrt[4]{64L + 15}}{4}$ is increasing, we get that the relation between $u_1$ and $u_2$ determines whether the sequence is either increasing, or decreasing for all $n$, i.e., if $u_1 < u_2$, then $u_n < u_{n+1}$ (for all $n$), and if $u_1 > u_2$ then $u_n > u_{n+1}$ (for all $n$). Now, the roots of the equation
\[
64x^3 - 64x - 15 = 0
\]
are $x_3 = -\frac{1}{8}$ and $x_1 = \frac{1 - \sqrt{61}}{8}$ and $x_3 = \frac{1 + \sqrt{61}}{8}$. So, we have several cases:

**Case 1.** If $u_1 = x_i$ for some $i = 1, 2, 3$, then $x_n = u_i$ for all $n$ and therefore, the limit is simply $u_i$ in this case.

**Case 2.** If $u_1 < x_1$, then $u_2 = f(u_1) < f(x_1) = x_1$ but also $u_2 > u_1$, which means that the sequence $\{u_n\}$ converges to $x_1$ in this case.

**Case 3.** If $x_1 < u_1 < x_2$ then $x_1 < u_n < x_2$ for all $n$ and moreover, $u_2 < u_1$ and so, $u_{n+1} < u_n$ for all $n$. Therefore, the sequence $\{u_n\}$ converges to $x_1$ in this case.

**Case 4.** If $x_2 < u_1 < x_3$ then $x_2 < u_n < x_3$ for all $n$ and moreover, $u_1 < u_2$ and so, $u_n < u_{n+1}$ for all $n$. Therefore, the sequence $\{u_n\}$ converges to $x_3$.

**Case 5.** If $x_3 < u_1$ then $x_3 < u_2 < u_1$ and so, $u_{n+1} < u_n$ for all $n$. In conclusion, $\{u_n\}$ converges to $x_3$. 