Problem 1. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function satisfying the relation:
\[
f(x+y+xy) = f(x) + f(y) + f(xy)
\]
for each \( x, y, \in \mathbb{R} \).
Prove that \( f(x+y) = f(x) + f(y) \) for each \( x, y \in \mathbb{R} \).

Solution. Letting \( x = y = 0 \) we obtain \( f(0) = 3f(0) \) and so, \( f(0) = 0 \). Then letting \( y = -1 \) (and \( x \) arbitrary) we obtain
\[
f(-1) = f(x) + f(-1) + f(-x),
\]
which yields \( f(-x) = -f(x) \) for all \( x \in \mathbb{R} \). Now, we simply replace \( x \) and \( y \) by \( -x \), respectively \(-y\) and obtain
\[
f(xy - x - y) = f(xy) + f(-x) + f(-y) = f(xy) - f(x) - f(y)
\]
which combined with the main relation yields
\[
f(xy - (x+y)) + f(xy + (x+y)) = 2f(xy).
\]
Now, for fixed \( xy = a \), we observe that \( x+y \) varies on the entire set of real numbers (i.e., it can be arbitrarily large and negative and also arbitrarily large and positive). This proves that for all \( a, b \in \mathbb{R} \) we have
\[
f(a-b) + f(a+b) = 2f(a).
\]
However, letting \( a = b \) in the above expression we get that
\[
f(0) + f(2a) = 2f(a)
\]
and so, \( f(2a) = 2f(a) \) because \( f(0) = 0 \).
Thus, \( f(a-b) + f(a+b) = f(2a) \) for all \( a, b \in \mathbb{R} \) which yields the relation asked in the problem.

Problem 2. Find all positive real numbers \( a \) with the property that the equation \( \log_a(x) - x = 0 \) has exactly one real solution.

Solution. We split our analysis into several cases:

Case 1. \( 0 < a < 1 \).
In this case, \( \log_a(x) \) decreases from \(+\infty\) to \(-\infty\), while \( x \) increases from 0 to \(+\infty\); so, using that \( f(x) := \log_a(x) - x \) is a continuous function (on \((0, +\infty)\)), then we conclude that for each \( a \in (0, 1) \) there exists a unique \( x \in (0, +\infty) \) such that \( f(x) = 0 \), i.e., \( \log_a(x) = x \).

Case 2. \( a > 1 \).
In this case the derivative of the above defined function \( f(x) \) is
\[
f'(x) = \frac{1}{x \cdot \ln(a)} - 1
\]
and so, \( f(x) \) is increasing on \((0, 1/\ln(a))\), while \( f(x) \) is decreasing on \((1/\ln(a), +\infty)\). We compute the global maximum of \( f(x) \) on \((0, +\infty)\):

\[
f\left(\frac{1}{\ln(a)}\right) = \frac{\ln\left(\frac{1}{\ln(a)}\right)}{\ln(a)} - \frac{1}{\ln(a)} = -\frac{\ln(\ln(a)) + 1}{\ln(a)}.
\]

Now, if the global maximum of \( f(x) \) is 0 then there exists indeed a single value of \( x \) for which \( \log_a(x) = x \); so,

**Subcase 2(i).** If \( a = e^{\frac{1}{2}} \) then there exists a unique value of \( x \) such that \( \log_a(x) = x \).

Now, if \( \ln(\ln(a)) + 1 > 0 \), then the global maximum of \( f(x) \) is negative and therefore,

**Subcase 2(ii).** If \( a > e^{\frac{1}{2}} \) then there exists no \( x \) such that \( \log_a(x) = x \).

Finally, if \( \ln(\ln(a)) + 1 < 0 \), then the global maximum of \( f(x) \) is positive and then we conclude that

**Subcase 2(iii).** If \( 1 < a < e^{\frac{1}{2}} \) then there exist exactly two values of \( x \) (one in the interval \((0, 1/\ln(a))\) and the other in \((1/\ln(a), +\infty)\)) since \( \lim_{x \to 0^+} f(x) = \lim_{x \to +\infty} f(x) = -\infty \) such that \( \log_a(x) = x \).

**Problem 3.**

(a) Find all integers \( n > 2 \) for which there exists an integer \( m \geq n \) such that \( m \) divides the least common multiple of \( m-1, m-2, \ldots, m-n+1 \).

(b) Find all positive integers \( n > 2 \) for which there exists exactly one integer \( m \geq n \) such that \( m \) divides the least common multiple of \( m-1, m-2, \ldots, m-n+1 \).

**Solution.** Let \( p^a \) be a prime power appearing in the prime power factorization of \( m \). Then \( m \) dividing \( \text{lcm}[m-1, \ldots, m-(n-1)] \) yields that \( p^a \) must divide one of the numbers \( m-i \) (for \( i = 1, \ldots, n-1 \)) and so, \( p^a \) must divide \( m-(m-i) = i \).

In conclusion, \( m \) divides \( \text{lcm}[m-1, \ldots, m-(n-1)] \) if and only if \( m \) divides \( \text{lcm}[1, \ldots, n-1] := L(n) \). So, the existence of at least one integer \( m \geq n \) with the property that it divides \( \text{lcm}[m-1, \ldots, m-(n-1)] \) is equivalent with asking that \( L(n) \geq n \). Now, since \( L(n) \geq (n-1)(n-2) \) and

\[
(n-1)(n-2) \geq n \quad \text{for all} \quad n \geq 4,
\]

while \( L(3) = 2 < 3 \) and \( L(2) = 1 < 2 \), we conclude that for all \( n \geq 4 \) there exists at least one integer \( m \) such that \( m \) divides \( \text{lcm}[m-1, \ldots, m-(n-1)] \).

Now, if we require that there exists precisely one integer \( m \geq n \) dividing \( \text{lcm}[m-1, \ldots, m-(n-1)] \) then we actually ask that there exists precisely one integer at least equal to \( n \) which divides \( L(n) \), i.e., that integer would be \( L(n) \). So, we’re asking in this case for which \( n \geq 4 \) we have that the only divisor of \( \text{lcm}[1, \ldots, n-1] \) at least equal to \( n \) is \( L(n) \). We claim that in this case we must have that \( n = 4 \).

First of all, we have \( L(4) = \text{lcm}[1,2,3] = 6 \) and so indeed only 6 is at least equal to 4 and divides 6. Now, if \( n \geq 5 \), then both \( (n-1)(n-2) \) and also \( (n-2)(n-3) \) are greater than \( n \) and they divide \( \text{lcm}[1, \ldots, n-1] \), which finishes our proof.

**Problem 4.** Find the minimum of

\[
\max\{a+b+c, b+c+d, c+d+e, d+e+f, e+f+g\}
\]
where the real numbers $a, b, c, d, e, f, g$ vary among all the possible nonnegative solutions to the equation $a + b + c + d + e + f + g = 1$.

*Solution.* We have that
\[
(a + b + c) + (d + e + f) + (e + f + g) \geq a + b + c + d + e + f + g = 1
\]
and therefore,
\[
M := \max\{a + b + c, b + c + d, c + d + e, d + e + f, e + f + g\} \geq \frac{1}{3}.
\]
On the other hand, this minimum value of $\frac{1}{3}$ for $M$ is attained in the case
\[
a = \frac{1}{3}, \quad b = c = 0, \quad d = \frac{1}{3}, \quad e = f = 0, \quad g = \frac{1}{3}.
\]