Problem 1. Consider the two sequences \( \{a_m\}_{m \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) defined by
\[
a_1 = 3 \quad \text{and for each } m \geq 1, \quad a_{m+1} = 3^{a_m}
\]
and
\[
b_1 = 100 \quad \text{and for each } n \geq 1, \quad b_{n+1} = 100^{b_n}.
\]
Find the smallest possible integer \( n \) such that \( b_n > a_{2019} \).

Solution. Clearly, \( b_1 > a_2 = 27 \) and then an easy induction yields that \( b_n > a_{n+1} \) for all \( n \geq 1 \). Next we prove the following (surprising) result.

Claim 0.1. For each \( n \geq 1 \), we have that \( b_n < a_{n+2} \).

Proof of Claim 0.1. Actually, we’ll prove by induction an even stronger claim:
\[
a_{n+2} > 2 + 5b_n \quad \text{for each } n \geq 1.
\]
Inequality (1) holds (easily) for \( n = 1 \) and then, using the inductive hypothesis, we get
\[
a_{n+3} = 3^{a_{n+2}} > 3^{2+5b_n} = 9 \cdot 243^{b_n} > 9 \cdot b_{n+1} > 2 + 5b_{n+1},
\]
as claimed. This concludes our proof of Claim 0.1.

Clearly, Claim 0.1 (coupled with the easy inequality \( b_n > a_{n+1} \)) yields that \( b_{2017} < a_{2019} < b_{2018} \) and so, the desired integer in this problem is \( 2018 \).

Problem 2. Let \( n > 1 \) be an integer and let \( a > 0 \) be a real number. Let \( x_1, \ldots, x_n \) be nonnegative real numbers satisfying: \( \sum_{i=1}^n x_i = a \). Find the maximum of \( \sum_{i=1}^{n-1} x_i x_{i+1} \).

Solution. Let \( x := \max_{1 \leq i \leq n} x_i \). Then
\[
\sum_{i=1}^{n-1} x_i x_{i+1} \leq x(a-x) \leq \frac{a^2}{4}
\]
with equality if (for example) \( x_1 = x_2 = \frac{a}{2} \).

Problem 3. Let \( N \) be the number of integer solutions to the equation \( x^3 - y^3 = z^5 - t^5 \) with the property that \( 0 \leq x, y, z, t \leq 2019^{2019} \). Let \( M \) be the number of integer solutions to the equation \( x^3 - y^3 = z^5 - t^5 + 1 \) with the property that \( 0 \leq x, y, z, t \leq 2019^{2019} \). Prove that \( N > M \).

Solution. For each \( 0 \leq i \leq 2019^{3 \cdot 2019} + 2019^{5 \cdot 2019} := L \), we let \( n_i \) be the number of integers \( 0 \leq a, b \leq 2019^{2019} \) with the property that \( a^3 + b^5 = i \). Then
\[
N = n_0^2 + n_1^2 + \cdots + n_L^2
\]
and \( M = n_0 n_1 + n_1 n_2 + \cdots + n_{L-1} n_L \). Then we see that
\[
N - M = \frac{n_0^2 + (n_0 - n_1)^2 + (n_1 - n_2)^2 + \cdots + (n_{L-1} - n_L)^2 + n_L^2}{2} > 0
\]
since \( n_0 = n_L = 1 \).

**Problem 4.** Find all \( n \in \mathbb{N} \) such that \( 2^8 + 2^{11} + 2^n \) is a perfect square.

**Solution.** If \( n \geq 8 \), then letting \( x := n - 8 \) then we need that
\[
(2^4)^2 \cdot (9 + 2^x)
\]
be a perfect square, which is equivalent with \( 9 + 2^x \) be a perfect square \( y^2 \). Thus
\[
2^x = (y - 3)(y + 3)
\]
and so, both \( y - 3 \) and \( y + 3 \) are powers of 2 which yields that the only possibility is

\[
y - 3 = 2^1 \quad \text{and} \quad y + 3 = 2^3,
\]
i.e., \( y = 5 \) and hence \( x = 4 \). So, \( n = 12 \); note that
\[
2^8 + 2^{11} + 2^{12} = 80^2.
\]
Now, if \( n < 8 \) then \( 2^8 + 2^{11} + 2^n \) is divisible by \( 2^n \) but not by \( 2^{n+1} \); thus \( n \) must be even. So, we only need to check \( n \in \{2, 4, 6\} \) and since
\[
1 + 2^6 + 2^0 = 577 \text{ is not a perfect square}
\]
\[
1 + 2^4 + 2^7 = 145 \text{ is not a perfect square}
\]
\[
1 + 2^2 + 2^5 = 37,
\]
we conclude that \( n = 12 \) is the only solution.