Problem 1. Find the maximum and the minimum possible value of the product $x_1 \cdot x_2 \cdots x_n$, where the real numbers $x_i$ satisfy the following properties:

- $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$; and
- $x_i \geq \frac{1}{n}$ for each $i = 1, \ldots, n$.

Solution. We use the inequality between the Arithmetic Mean and the Geometric Mean and therefore, conclude that

$$\sqrt[\frac{1}{n}]{x_1^2 \cdot x_2^2 \cdots x_n^2} \leq \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} = \frac{1}{n}$$

with equality if and only if $x_1^2 = x_2^2 = \cdots = x_n^2 = \frac{1}{n}$. So, the maximum of $\prod_{i=1}^{n} x_i$ is $\frac{1}{n^n}$ and it is attained when $x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt{n}}$ (which is allowed since $\frac{1}{\sqrt{n}} > \frac{1}{n}$).

Now, in order to determine the minimum of $\prod_{i=1}^{n} x_i$ we use the following easy claim.

Claim 0.1. Let $u, v, u_1, v_1$ be positive real numbers such that

- $u + v = u_1 + v_1$; and
- $\min\{u, v\} \geq u_1$.

Then $uv \geq u_1v_1$.

Proof of Claim 0.1. Using the above hypothesis, we have that $v_1 \geq u_1$; also, without loss of generality, we may assume $v \geq u$. Then (because $u_1 \leq u \leq v \leq v_1$) we have

$$v_1 - u_1 \geq v - u$$

and so, $4u_1v_1 = (u_1 + v_1)^2 - (v_1 - u_1)^2 \leq (u + v)^2 - (v - u)^2 = 4uv$, thus proving the desired claim. \hfill \square

So, when we minimize $\prod_{i=1}^{n} x_i^2$ with $x_n = \max_{i=1}^{n} x_i$, then for any $i = 1, \ldots, n-1$, we may replace $x_1$ by $x'_1 := \frac{1}{n}$ and then replace $x_n$ by $x'_n := \sqrt{x_1^2 + x_2^2 + \cdots + \frac{1}{n^2}}$—this will only decrease the above product. So, the minimum of $\prod_{i=1}^{n} x_i$ is obtained for

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{n} \quad \text{and} \quad x_n = \sqrt{\frac{n^2 - n + 1}{n^2}}$$

and so, $\min\prod_{i=1}^{n} x_i = \frac{\sqrt{n^2 - n + 1}}{n^n}$.

Problem 2. We let $f : [0, 1) \rightarrow [0, 1)$ be defined by the properties:

$$f(x) = \begin{cases} 
\frac{f(2x)}{2} & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{3 + f(2x-1)}{4} & \text{if } \frac{1}{2} \leq x < 1 
\end{cases}$$
Find $f(x)$ for each $x \in [0,1)$; you may express your answer in terms of the expansion of $x$ in base 2.

**Solution.** We write $x = 0.b_1 b_2 \cdots b_n \cdots$ in base 2, i.e., $b_n \in \{0,1\}$ for all $n \geq 1$. Also, we use the convention that we write

$$0.b_1 \cdots b_n 100 \cdots \cdots$$

and not $0.b_1 \cdots b_n 011 \cdots 1 \cdots$. Then our definition for $f(x)$ yields the following:

- if $b_1 = 0$ then $f(x) = \frac{f(0.b_2 b_3 \cdots b_n \cdots)}{4}$, while
- if $b_1 = 1$ then $f(x) = \frac{3 + f(0.b_2 b_3 \cdots b_n \cdots)}{4}$.

In both cases, we get

$$f(x) = \frac{b_1}{2} + \frac{b_1}{4} + \frac{b_2}{8} + \frac{b_2}{16} + \frac{f(0.b_3 b_4 \cdots b_n \cdots)}{16}$$

and then inductively, we obtain

$$f(x) = \frac{b_1}{2} + \frac{b_1}{4} + \frac{b_2}{8} + \frac{b_2}{16} + \frac{f(0.b_3 b_4 \cdots b_n \cdots)}{16}$$

and furthermore, for any $m \geq 1$, we have

$$f(x) = 0.b_1 b_2 b_3 \cdots b_{m-1} b_{m-1} b_m b_m + \frac{f(0.b_{m+1} b_{m+2} \cdots)}{4^m}.$$

Now, since $f(z) < 1$ for each $z$, then

$$\frac{f(0.b_{m+1} b_{m+2} \cdots)}{4^m} \to 0 \text{ as } m \to \infty$$

and so, we conclude that

$$f(0.b_1 b_2 \cdots b_n \cdots) = 0.b_1 b_2 b_3 \cdots b_n b_n \cdots$$

**Problem 3.** Find all real numbers $a$ for which there exist nonnegative real numbers $x_1, \ldots, x_5$ satisfying the following property:

$$\sum_{k=1}^{5} k^{2i-1} \cdot x_k = a^i \text{ for each } i = 1, 2, 3.$$

**Solution.** The given relations yield

$$\sum_{k=1}^{5} (a - k^3) \cdot x_k = 0 \text{ and } \sum_{k=1}^{5} (ak^3 - k^5) x_k = 0.$$

So,

$$\sum_{1 \leq k \leq 5} \sum_{1 \leq k \leq 5} (a - k^2) k x_k = \sum_{1 \leq k \leq 5} (k^2 - a) k x_k$$

and

$$\sum_{1 \leq k \leq 5} (a - k^2) k^3 x_k = \sum_{1 \leq k \leq 5} (k^2 - a) k^3 x_k.$$
However,

\[ \sum_{1 \leq k \leq 5} (a - k^2)k^3 x_k \]

\[ \leq \sum_{1 \leq k \leq 5} (a - k^2) \cdot a \cdot k x_k \]

\[ = \sum_{1 \leq k \leq 5} (k^2 - a) k x_k \cdot a \]

\[ \leq \sum_{1 \leq k \leq 5} (k^2 - a) k^3 x_k \]

and since the first and the last of these sums are equal, then it must be that both inequalities above are actually equalities. Now, if \( a \notin \{1^2, 2^2, 3^2, 4^2, 5^2\} \) then the above inequalities cannot become equalities (since not all of the \( x_k \) can be equal to zero). On the other hand, for each \( m \in \{1, \ldots, 5\} \), if \( a = m^2 \) then letting \( x_k = 0 \) if \( k \neq m \) while \( x_m = m \), then all of the hypotheses are met.

**Problem 4.** Let \( m \in \mathbb{N} \) and let \( a_1, \ldots, a_m \in \mathbb{N} \). Prove that there exists a positive integer \( n < 2^m \) and there exist positive integers \( b_1, \ldots, b_n \) satisfying the following properties:

(i) for any two distinct subsets \( I, J \subseteq \{1, \ldots, n\} \), we have that \( \sum_{k \in I} b_k \neq \sum_{k \in J} b_k \); and

(ii) for each \( i = 1, \ldots, m \), there exists a subset \( J_i \subseteq \{1, \ldots, n\} \) such that \( a_i = \sum_{k \in J_i} b_k \).

**Solution.** We write for each \( i = 1, \ldots, m \):

\[ a_i = \sum_{j \in M_i} 2^j, \]

where \( M_i \) is the set of nonnegative integers corresponding to the positions in the writing of \( a_i \) in base 2 where the digit of \( a_i \) equals 1. Writing similarly each \( b_i \) (for \( 1 \leq i \leq n \))

\[ b_i = \sum_{j \in A_i} 2^j, \]

then the above conditions (i)-(ii) are satisfied if the following conditions are met:

(i) \( A_i \cap A_j = \emptyset \) if \( 1 \leq i < j \leq n \); and

(ii) for each \( i = 1, \ldots, m \), there exists \( J \subseteq \{1, \ldots, n\} \) such that \( M_i = \cup_{j \in J} A_j \).

Furthermore, we need \( n \leq 2^m - 1 \).

Now, we prove the existence of such sets \( \{A_j\}_{1 \leq j \leq n} \) corresponding to given sets \( \{M_i\}_{1 \leq i \leq m} \) (with \( n \leq 2^m - 1 \)). We argue by induction on \( m \); the case \( m = 1 \) is immediate since we may take \( n = 1 < 2^1 \) and \( A_1 := M_1 \).

Next, we assume we constructed \( A_1, \ldots, A_n \) (with \( n \leq 2^m - 1 \)) corresponding to the sets \( M_1, \ldots, M_m \) and given a new set \( M_{m+1} \), we construct the following
sets (note that if one of these sets $A'_i$ is empty, then we could simply disregard the corresponding $a'_i$):

$$A'_{2n+1} := M_{m+1} \setminus \left( \bigcup_{i=1}^{n} A_i \right)$$

$$A'_2 := M_{m+1} \cap A_i \text{ for } 1 \leq i \leq n$$

$$A'_{2i-1} := A_i \setminus M_{m+1} \text{ for } 1 \leq i \leq n.$$  

Since $A_i = A'_{2i} \cup A'_{2i-1}$ for $1 \leq i \leq n$, then each $M_j$ (for $1 \leq j \leq m$) can be written as a union of the sets $A'_i$ (for $1 \leq i \leq 2n$). Also, since $A_i \cap A_j = \emptyset$ if $1 \leq i < j \leq n$, then $A'_i \cap A'_j = \emptyset$ if $1 \leq i < j \leq 2n$. Furthermore,

$$M_{m+1} = \bigcup_{i=1}^{2n+1} A'_i$$

and also, $A'_{2n+1}$ is disjoint from each $A'_i$ for $1 \leq i \leq 2n$. So, the hypotheses are met for the sets $A'_i$ for $1 \leq i \leq 2n+1$ and clearly,

$$2n + 1 \leq 2(2^n - 1) + 1 = 2^{m+1} - 1.$$