**PUTNAM PRACTICE SET 8**

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**Problem 1.** Prove that if $a, b, c \in \mathbb{C}$ and the following relations are satisfied:
- $a + b + c = 0$; and
- $|a| = |b| = |c|$, 
then $a^3 = b^3 = c^3$.

Can this result be extended to more than 3 complex numbers?

**Problem 2.** If the series $\sum_{n=1}^{\infty} a_n$ of real numbers converges, does $\sum_{n=1}^{\infty} a_n^3$ converge?

**Problem 3.** For what pairs $(a, b)$ of positive real numbers we have that the integral
\[
\int_{a}^{b} \left( \sqrt{\sqrt{x} + a - \sqrt{x} - \sqrt{\sqrt{x} - \sqrt{x} - b}} \right) \, dx
\]
converges.

**Problem 4.** For each $n \in \mathbb{N}$, we let $S_n$ be the set of all pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with the property that $x^3 - 3xy^2 + y^3 = n$.

(a) For each $n \in \mathbb{N}$, prove that either $S_n$ is the empty set, or it has at least 3 elements.

(b) Prove that $S_{2021}$ is the empty set.