Problem 1. Prove that for each positive integer $n$, we have
\[
\left( \frac{2n - 1}{e} \right)^{\frac{2n-1}{2}} < \prod_{i=1}^{n}(2i - 1) < \left( \frac{2n + 1}{e} \right)^{\frac{2n+1}{2}},
\]
where $e$ is base of the natural logarithm.

Problem 2. For any square matrix $A$ with real entries, we can define
\[
\sin(A) := \sum_{n=0}^{\infty} \frac{(-1)^n A^{2n+1}}{(2n+1)!},
\]
i.e., the above series converges. Determine with proof whether there exists some matrix $A$ with real entries such that
\[
\sin(A) = \begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix}.
\]

Problem 3. Let $P \in \mathbb{R}[x]$ with the property that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ such that $P(x) = Q_1(x)^2 + Q_2(x)^2$.

Problem 4. Let $a_n$ be real numbers so that the following power series expansion holds:
\[
\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.
\]
Prove that for each integer $n \geq 0$, there exists a positive integer $m$ such that $a_{n+1}^2 + a_n^2 = a_m$. 