1 Spectral decomposition and relaxation time

Let $X$ be a reversible Markov chain on the finite state space $S$ with transition matrix $P$ and invariant distribution $\pi$. Let $f, g : S \to \mathbb{R}$. Their inner product is defined to be

$$\langle f, g \rangle_\pi = \sum_x f(x)g(x)\pi(x).$$

**Theorem 1.1.** Let $P$ be reversible with respect to $\pi$. The inner product space $(\mathbb{R}^S, \langle \cdot, \cdot \rangle_\pi)$ has an orthonormal basis of real-valued eigenfunctions $(f_j)_{j \leq |S|}$ corresponding to real eigenvalues $(\lambda_j)$ and the eigenfunction $f_1$ corresponding to $\lambda_1 = 1$ can be taken to be the constant vector $(1, \ldots, 1)$. Moreover, the transition matrix $P^t$ can be decomposed as

$$P^t(x, y) = \pi(y) = 1 + \sum_{j=2}^{|E|} f_j(x)f_j(y)\lambda_j^t.$$

**Proof.** We consider the matrix $A(x, y) = \sqrt{\pi(x)}P(x, y)/\sqrt{\pi(y)}$ which using reversibility of $P$ is easily seen to be symmetric. Therefore, we can apply the spectral theorem for symmetric matrices and get the existence of an orthonormal basis $(g_j)$ corresponding to real eigenvalues. It is easy to check that $\sqrt{\pi}$ is an eigenfunction of $A$ with eigenvalue $1$. Let $D$ be the diagonal matrix with elements $(\sqrt{\pi(x)})$. Then $A = DP^{-1}$ and it is easy to check that $f_j = D^{-1}g_j$ are eigenfunctions of $P$ and $(f_j, f_i)_\pi = 1(i = j)$. So we have $P^t f_j = \lambda_j^t f_j$ and hence

$$P^t(x, y) = (P^t 1_y)(x) = \sum_{j=1}^{|S|} \lambda_j^t f_j(x)f_j(1_y)_\pi = \sum_{j=1}^{|S|} \lambda_j^t f_j(x)f_j(y)\pi(y).$$

Using that $f_1 = 1$ and $\lambda_1 = 1$ gives the desired decomposition. \qed

Let $P$ be a reversible matrix with respect to $\pi$. We order its eigenvalues

$$1 = \lambda_1 > \lambda_2 \geq \ldots \lambda_{|S|} \geq -1.$$  

We let $\lambda_* = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P, \lambda \neq 1\}$ and define $\gamma_* = 1 - \lambda_*$ to be the absolute spectral gap. The spectral gap is defined to be $\gamma = 1 - \lambda_2$.

**Exercise 1.2.** Check that if the chain is lazy then $\gamma_* = \gamma$.

**Definition 1.3.** The relaxation time for a reversible Markov chain is defined to be

$$t_{rel} = \frac{1}{\gamma_*}.$$  

Let $f : S \to \mathbb{R}$. We write

$$E_\pi[f] = \sum_x f(x)\pi(x) \quad \text{and} \quad \text{Var}_\pi(f) = E_\pi[(f - E_\pi[f])^2].$$

**Exercise 1.4 (Poincaré inequality).** Let $P$ be a lazy and reversible matrix with respect to the invariant distribution $\pi$. Then for all $f : S \to \mathbb{R}$ and all $t \geq 0$

$$\text{Var}_\pi(P^t f) \leq e^{-2t/t_{rel}}\text{Var}_\pi(f).$$

(Hint: Use the spectral theorem)