Recall (Oliveira, Peres – S.)

\[ 0 < \alpha \leq \frac{1}{2} \exists \ c_{\alpha} \text{ and } c_{\alpha}' > 0 \text{ s.t. } \forall \text{ finite reversible lazy MC's} \]

\[ c_{\alpha}' \cdot \text{th}(\alpha) \leq \text{twix} \leq c_{\alpha} \cdot \text{th}(\alpha) \]

Recall \[ \text{th}(\alpha) = \max_{x, A: \pi(A) > 0} \mathbb{E}_x[T_A] \]

**Removing the reversibility assumption**

For all integers \[ \mathbb{Z}_n \]

\[ \text{twix} \times n^2 \quad \text{and} \quad \text{th}(\alpha) \asymp n + o(n) \]

\[ \max_{x, y} \mathbb{E}_x[T_y] \asymp n \]

For \[ \alpha \in (0, 1) \] define \[ A(\alpha) = \{ A = (A_t)_{t \geq 0} : \pi(A_t) \geq \alpha, \forall t \geq 0 \} \]

and define also \[ \text{to} = \inf\{ t \geq 0 : x \not\in A_t \} \text{ for } A = (A_t)_{t \geq 0} \]

**Theorem (Winkler – S.)**

Fix \[ \alpha < \frac{1}{2} \]. Then there exist two positive constants \[ c_{\alpha} \text{ and } c_{\alpha}' \] s.t.

For all irreducible finite Markov chains \( \Phi \)

\[ \text{twom}(\alpha) = \sup_{x, A \in A(\alpha)} \mathbb{E}_x[T_A] \text{ then} \]

\[ c_{\alpha}' \cdot \text{twom}(\alpha) \leq \text{twix} \leq c_{\alpha} \cdot \text{twom}(\alpha) \]

**Hitting times** (Griffiths, Kang, Oliveira, Patel)

**Theorem 1** Let \( 0 < \alpha < \beta < \frac{1}{2} \). Then for every irreducible finite Markov chain

\[ \text{th}(\alpha) < \text{th}(\beta) + (\frac{1 - \alpha}{\alpha}) \cdot \text{th}(1 - \beta) < \frac{1}{\alpha} \cdot \text{th}(\beta) \]

(Recall \[ \text{th}(\alpha) = \max_{x, A: \pi(A) > 0} \mathbb{E}_x[T_A] \])

**Remark** These 2 ineq. are sharp, i.e. \( 0 < \alpha < \beta < \frac{1}{2} \) there exists an irred. finite Markov chain for which all three terms are equal.

\( \beta = \frac{1}{2} \) is a boundary case, in the sense that \( \forall \beta > \frac{1}{2} \) there exists a class of irred. finite MC's s.t. \( \text{th}(\alpha)/\text{th}(\beta) \) can be made arbitrarily large.
**Lemma 1** For an irreducible, finite MC for any \( \phi \neq A, B \in S \) state space

Define \( d^+(A, B) = \max_{x \in A} E_x[T_B] \) and \( d^-(A, B) = \min_{x \in A} E_x[T_B] \).

Then \( \pi(A) \leq \frac{d^+(A, B)}{d^+(A, B) + d^-(B, A)} \).

Non-rigorous explanation: \( \pi(A) \cdot (d^+(A, B) + d^-(B, A)) \leq d^+(A, B) \).

Let \( x \in A \) be s.t. \( E_x[T_B] = d^+(A, B) \).

\[ \pi(A) \cdot (E_x[T_B] + d^-(B, A)) \leq d^+(A, B) \]

Ergodic theorem \( \Rightarrow \) by time \( t \) the chain visits the set \( A \) : \( \pi(A) \cdot t \) times

Define \( \tau_{B,A} = \min \{ t \geq \tau_B : X_t \in A \} \).

By time \( \tau_{B,A} \) "the chain spends time \( \pi(A) E_x[T_{B,A}] \)" in \( A \).

Also the time spent \( \leq E_x[T_B] \) (after \( \tau_B \) no more visits to \( A \)).

\[ \Rightarrow \pi(A) E_x[T_{B,A}] \leq E_x[T_B] = d^+(A, B) \]

\[ \geq E_x[T_B] + d^-(B, A) \]

**Proof of Theorem 1** Fix \( x \) and a set \( A \) with \( \pi(A) \geq \alpha \).

Want to show \( E_x[T_A] \leq h_{(\beta)} + (\frac{1}{\alpha} - 1) h_{(1-\beta)} \).

Want to define a set \( B \) with \( \pi(B) \geq \beta \) so that we first wait to hit \( B \) and then starting from there the time to hit is controlled by the second term.

Define \( B = \{ y : E_y[T_A] \leq (\frac{1}{\alpha} - 1) h_{(1-\beta)} \} \).

If we show \( \pi(B) \geq \beta \), then we are done, because

\[ E_x[T_A] \leq E_x[T_B] + \max_{y \in B} E_y[T_A] \leq h_{(\beta)} + (\frac{1}{\alpha} - 1) h_{(1-\beta)} \]

Claim \( \pi(B) \geq \beta \).

Suppose not, i.e. \( \pi(B) < \beta \) and let \( C = B^c \). Then \( \pi(C) > 1-\beta \).

\[ \pi(A) \leq \frac{d^+(A, C)}{d^+(A, C) + d^-(C, A)} \]

using Lemma.
Proof of Lemma 1

Lemma 2 Let \( X \) be an irreducible finite Markov chain with values in \( S \). Let \( \mu \) be a prob. distribution and \( \tau \) a stopping time s.t.
\[
\mathbb{P}_\mu(X_\tau = x) = \mu(x) \quad \forall x.
\]
Then
\[
\mathbb{E}_\mu \left[ \sum_{i=0}^{\tau-1} 1(X_i \in A) \right] = \pi(A), \quad \mathbb{E}_\mu[\tau], \quad \forall A \subseteq S.
\]

Proof Exercise

Define \( \mathbb{V}(x) = \mathbb{E}_{X_\tau} \left[ \sum_{i=0}^{\tau-1} 1(X_i = x) \right] \Rightarrow \mathbb{V} = \mathbb{V}P \Rightarrow \mathbb{V} \) has to be a multiple of \( \pi \).

Proof of L1 Define
\[
\tau_{BA} = \min \{ t > \tau_B : X_t \in A \}.
\]
and an auxiliary Markov chain with transition matrix \( Q \)
\[
Q(x,y) = \mathbb{P}_x(X_{\tau_{BA}} = y), \quad x,y \in A.
\]
This is a finite irreducible MC \( \Rightarrow \) it possesses an invariant distr. \( \mu \).

Let \( \nu(y) = \mathbb{P}_\mu(X_{\tau_B} = y), \quad y \in B \). Call \( \tau = \tau_{BA} \)

Count the number of visits to \( A \) up until \( \tau_{BA} \), starting from \( \mu \).
\[
\mathbb{E}_\mu \left[ \sum_{i=0}^{\tau-1} 1(X_i \in A) \right] \leq \mathbb{E}_\mu[\tau_B] \quad \star
\]
Because \( \mu \) is invariant for Q \( \Rightarrow \mathbb{P}_\mu(X_\tau = x) = \mu(x) \quad \forall x \). So the conditions of Lemma 2 are satisfied
\[
\Rightarrow \mathbb{E}_\mu \left[ \sum_{i=0}^{\tau-1} 1(X_i \in A) \right] = \pi(A) \mathbb{E}_\mu[\tau] = \pi(A) \left( \mathbb{E}_\mu[\tau_B] + \mathbb{E}_\mu[\tau_A] \right)
\]
So
\[
\pi(A) \left( \mathbb{E}_\mu[\tau_B] + \mathbb{E}_\mu[\tau_A] \right) \leq \mathbb{E}_\mu[\tau_B] \quad \star
\]
\[
\Rightarrow \pi(A) \mathbb{E}_\mu[\tau_A] \leq (1-\pi(A)) \mathbb{E}_\mu[\tau_B] .
\]

This now completes the proof, because

\[ \pi(A) \leq \frac{d^+(A, B)}{d^+(A, B) + d^-(B, A)} = \pi(A) d^-(B, A) \leq (1 - \pi(A)) d^+(A, B) \]

and \( \mathbb{E}_\mu[\tau_A] \geq d^-(B, A) \) and \( \mathbb{E}_\nu[\tau_B] \leq d^+(A, B) \)

(\( \nu \) is supported on B and \( \mu \) is supported on A). \( \Box \)