Critical percolation

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Percolation - definitions

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This minicourse will focus on recent advances around this problem, with particular emphasis on the growing understanding of the importance of the Aizenman-Kesten-Newman argument. (but we will only get to it in the second hour)
Theorem

\[ E_{pc}(|C(0)|) = \infty. \]
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\[ \mathbb{E}_{p_c}(|\mathcal{C}(0)|) = \infty. \]

Proof.

Fix \( p \) and denote \( \chi = \mathbb{E}_p(|\mathcal{C}(0)|) \). Let

\[ \varepsilon < \frac{1}{4d\chi}. \]
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**Theorem**

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We will show that at \( p + \varepsilon \) there is no infinite cluster. Consider \( p + \varepsilon \) percolation as if we take \( p \)-percolation and then “sprinkle” each edge with probability \( \varepsilon \). For a vertex \( x \) and a sequence of directed edges \( e_1, \ldots, e_n \), denote by \( E_{x,e_1,\ldots,e_n} \) the event that 0 is connected to \( x \) by a path \( \gamma_1 \) in \( p \)-percolation from 0 to \( e_1^- \) then \( e_1 \) is sprinkled, then there is a path \( \gamma_2 \) from \( e_1^+ \) to \( e_2^- \) then \( e_2 \) is sprinkled and so on. We end with a path \( \gamma_{n+1} \) from \( e_n \) to \( x \). We require all the \( \gamma_i \) to be disjoint. Clearly \( 0 \leftrightarrow x \) is \( p + \varepsilon \) percolation if and only if there exist some \( e_1, \ldots, e_n \) (possibly empty) such that \( E_{x,e_1,\ldots,e_n} \) hold.
Theorem

$\mathbb{E}_{p_c}(|C(0)|) = \infty.$
Theorem

\[ \mathbb{E}_{p_c}(|\mathcal{C}(0)|) = \infty. \]

Proof.

\[ \chi = \mathbb{E}_p(|\mathcal{C}(0)|), \quad \varepsilon < 1/4d\chi, \quad E_{x,e_1,\ldots,e_n} \text{ is the event that } \exists \gamma_i \text{ from } e^+_i \text{ to } e^-_i, \text{ disjoint, and all } e_i \text{ are sprinkled.} \]

\[ \mathbb{P}_{p+\varepsilon}(0 \leftrightarrow x) \leq \sum_{n=0}^{\infty} \sum_{e_1,\ldots,e_n} \mathbb{P}(E_{x,e_1,\ldots,e_n}). \]
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Proof.

$\chi = \mathbb{E}_p(|\mathcal{C}(0)|)$, $\varepsilon < 1/4d\chi$, $E_{x,e_1,\ldots,e_n}$ is the event that $\exists \gamma_i$ from $e_{i-1}^+$ to $e_i^-$, disjoint, and all $e_i$ are sprinkled.

$$\mathbb{P}_{p+\varepsilon}(0 \leftrightarrow x) \leq \sum_{n=0}^{\infty} \sum_{e_1,\ldots,e_n} \mathbb{P}(E_{x,e_1,\ldots,e_n}).$$

By the BK inequality

$$\leq \sum_{n=0}^{\infty} \sum_{e_1,\ldots,e_n} \mathbb{P}_p(0 \leftrightarrow e_1^-)\mathbb{P}_p(e_1^+ \leftrightarrow e_2^-) \cdots \mathbb{P}(e_n^+ \leftrightarrow x)\varepsilon^n$$
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**Proof.**

\[ \chi = \mathbb{E}_p(|\mathcal{C}(0)|), \quad \varepsilon < 1/4d\chi, \quad E_{x,e_1,...,e_n} \text{ is the event that } \exists \gamma_i \text{ from } e^+_i \text{ to } e^-_i, \text{ disjoint, and all } e_i \text{ are sprinkled.} \]

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By the BK inequality

\[ \leq \sum_{n=0}^{\infty} \sum_{e_1,...,e_n} \mathbb{P}_p(0 \leftrightarrow e^-_1)\mathbb{P}_p(e^+_1 \leftrightarrow e^-_2) \cdots \mathbb{P}_p(e^+_n \leftrightarrow x)\varepsilon^n \]

Summing over all \( x \) gives

\[ \chi(p+\varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x,e_1,...,e_n} \mathbb{P}_p(0 \leftrightarrow e^-_1)\mathbb{P}_p(e^+_1 \leftrightarrow e^-_2) \cdots \mathbb{P}_p(e^+_n \leftrightarrow x). \]
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Proof.
\[ \chi(p+\varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x,e_1,...,e_n} P_p(0 \leftrightarrow e_1^-)P_p(e_1^+ \leftrightarrow e_2^-) \cdots P_p(e_n^+ \leftrightarrow x). \]

Summing over \( x \) gives one \( \chi(p) \) term which we can take out of the sum
\[ = \sum_{n=0}^{\infty} \varepsilon^n \chi(p) \sum_{e_1,...,e_n} P_p(0 \leftrightarrow e_1^-)P_p(e_1^+ \leftrightarrow e_2^-) \cdots P_p(e_{n-1}^+ \leftrightarrow e_n^-). \]
Theorem

$E_{p_c}(|\mathcal{C}(0)|) = \infty$.

Proof.

$\chi(p + \varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x, e_1, \ldots, e_n} P_p(0 \leftrightarrow e_1^-) P_p(e_1^+ \leftrightarrow e_2^-) \cdots P_p(e_n^+ \leftrightarrow x)$.

Summing over $x$ gives one $\chi(p)$ term which we can take out of the sum

$= \sum_{n=0}^{\infty} \varepsilon^n \chi(p) \sum_{e_1, \ldots, e_n} P_p(0 \leftrightarrow e_1^-) P_p(e_1^+ \leftrightarrow e_2^-) \cdots P_p(e_{n-1}^+ \leftrightarrow e_n^-)$.

$e_n^+$ has $2d$ possibilities. Summing over $e_n^-$ gives another $\chi$ term. Taking both out of the sum gives

$= \sum_{n=0}^{\infty} \varepsilon^n \cdot 2d \chi(p)^2 \sum_{e_1, \ldots, e_{n-1}} P_p(0 \leftrightarrow e_1^-) \cdots P_p(e_{n-2}^+ \leftrightarrow e_{n-1}^-)$. 
Theorem

\[ \mathbb{E}_{p_c}(|\mathcal{C}(0)|) = \infty. \]

Proof.

\[ \chi(p) = \mathbb{E}_p(|\mathcal{C}(0)|), \varepsilon < 1/4d\chi(p), \]

\[ \chi(p + \varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x,e_1,\ldots,e_n} \mathbb{P}_p(0 \leftrightarrow e_1^{-})\mathbb{P}_p(e_1^{+} \leftrightarrow e_2^{-}) \cdots \mathbb{P}_p(e_n^{+} \leftrightarrow x) \]

\[ = \sum_{n=0}^{\infty} \varepsilon^n \cdot (2d)^n \chi(p)^{n+1} \]
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\[ = \sum_{n=0}^{\infty} \varepsilon^n \cdot (2d)^n \chi(p)^{n+1} < \infty. \]
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This shows that \( p + \varepsilon \leq p_c. \)
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This shows that \( p + \varepsilon \leq p_c. \) The theorem is then proved by contradiction.

The argument also gives

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\chi(p) \geq \frac{1}{4d(p_c - p)} \quad \forall p < p_c.
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Proof.
\[ \chi(p) = \mathbb{E}_p(\vert \mathcal{C}(0) \vert), \ \varepsilon < 1/4d\chi(p), \]
\[ \chi(p + \varepsilon) \leq \sum_{n=0}^{\infty} \varepsilon^n \sum_{x,e_1,\ldots,e_n} \mathbb{P}_p(0 \leftrightarrow e^-_1)\mathbb{P}_p(e^+_1 \leftrightarrow e^-_2) \cdots \mathbb{P}_p(e^+_n \leftrightarrow x) \]
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The argument also gives
\[ \chi(p) \geq \frac{1}{4d(p_c - p)} \quad \forall p < p_c. \]

This is sharp on a tree but not in general.
For a set $S \subset \mathbb{Z}^d$ denote by $\partial S$ the set of $x \in S$ with a neighbour $y \not\in S$.

**Theorem**

*Let $S \subset \mathbb{Z}^d$ be some finite set containing 0. Then*

$$\sum_{x \in \partial S} \mathbb{P}_{p_c}(0 \leftrightarrow^S x) \geq 1.$$
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**Proof sketch.**

Let $x \in \mathbb{Z}^d$. If $0 \leftrightarrow x$ then there exists $0 = y_1, \ldots, y_n = x$ such and open paths $\gamma_i$ such that

- $\gamma_i$ is from $y_i$ to $y_{i+1}$ and is contained in $y_i + S$. 
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2. The $\gamma_i$ are disjoint.
For a set $S \subset \mathbb{Z}^d$ denote by $\partial S$ the set of $x \in S$ with a neighbour $y \notin S$.

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And we have $n \geq r|x|$ for some number $r > 0$ that depends on $S$. 
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And we have $n \geq r|x|$ for some number $r > 0$ that depends on $S$. A calculation similar to the previous proof shows that

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\mathbb{P}(0 \leftrightarrow x) \leq \sum_{n \geq r|x|} \left( \sum_{y \in \partial S} \mathbb{P}_{p_c}(0 \leftrightarrow y) \right)^n.
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\mathbb{P}(0 \leftrightarrow x) \leq \sum_{n \geq r|x|} \left( \sum_{y \in \partial S} \mathbb{P}_{p_c}(0 \leftrightarrow_S y) \right)^n.
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If the value in the parenthesis is smaller than 1 then \( \mathbb{P}(0 \leftrightarrow x) \) decays exponentially in \( |x| \), contradicting the previous theorem.
Theorem

Let $S \subset \mathbb{Z}^d$ be some finite set containing 0. Then
$$\sum_{x \in \partial S} \mathbb{P}_{p_c} (0 \leftrightarrow x \mid S) \geq 1.$$ 

A full proof can be found in H. Duminil-Copin and V. Tassion, *A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$*, L’Enseignement Mathématique, 62(1/2) (2016), 199-206.
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Theorem (Menshikov||Aizenman-Barsky)

*For any $p < p_c$, $\chi(p) < \infty$.*
Theorem

Let \( S \subset \mathbb{Z}^d \) be some finite set containing 0. Then
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Theorem (Menshikov∥Aizenman-Barsky)

For any \( p < p_c \) \( \chi(p) < \infty \).

(recall that \( \chi(p) = \mathbb{E}_p(|\mathcal{C}(0)|) \) and that what we proved before is \( \chi(p_c) = \infty \)).
Theorem

Let $S \subset \mathbb{Z}^d$ be some finite set containing 0. Then
\[ \sum_{x \in \partial S} \mathbb{P}_{p_c}(0 \leftrightarrow x) \geq 1. \]

Two applications:

Lemma (K-Nachmias, 2011)

For any $x \in \partial \Lambda_n$, $\Lambda_n := [-n, n]^d$,
\[ \mathbb{P}_{p_c}(0 \leftrightarrow \Lambda_n x) \geq c \exp(-C \log^2 n). \]

Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$,
\[ \mathbb{P}_{p_c}(x \leftrightarrow \Lambda_{2n} y) \geq cn^{-C}. \]

All constants $c$ and $C$ might depend on the dimension.
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xrightarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. 
Lemma (Cerf, 2015)

For any \( x, y \in \Lambda_n \), \( \mathbb{P}_{pc}(x \xleftarrow{\Lambda_{2n}} y) \geq cn^{-C} \).

Proof.

Assume first that \( x - y = (2k, 0, \ldots, 0) \), \( k \leq n \). By the theorem there exists a \( z \in \partial \Lambda_k \) such that

\[
\mathbb{P}(0 \xleftarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial \Lambda_k|}
\]
Lemma (Cerf, 2015)

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Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. By the theorem there exists a $z \in \partial \Lambda_k$ such that

$$\mathbb{P}(0 \xleftarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial \Lambda_k|} \geq \frac{c}{k^d-1}.$$
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Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. By the theorem there exists a $z \in \partial \Lambda_k$ such that

$$\mathbb{P}(0 \xrightarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial \Lambda_k|} \geq \frac{c}{k^d - 1}.$$

By rotation and reflection symmetry we may assume $z$ is in some face of $\Lambda_k$, for example $z_1 = k$. 
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xleftarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. By the theorem there exists a $z \in \partial \Lambda_k$ such that

$$\mathbb{P}(0 \xleftarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial \Lambda_k|} \geq \frac{c}{k^{d-1}}.$$

By rotation and reflection symmetry we may assume $z$ is in some face of $\Lambda_k$, for example $z_1 = k$. Let $\overline{z}$ be the reflection of $z$ in the first coordinate i.e. $\overline{z} = (-z_1, z_2, \ldots, z_d)$.
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \overset{\Lambda_{2n}}{\leftrightarrow} y) \geq cn^{-C}$.

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Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. By the theorem there exists a $z \in \partial \Lambda_k$ such that

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Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xrightarrow{\Lambda_2 n} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. By the theorem there exists a $z \in \partial \Lambda_k$ such that

$$\mathbb{P}(0 \xrightarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial \Lambda_k|} \geq \frac{c}{k^{d-1}}.$$ 

By rotation and reflection symmetry we may assume $z$ is in some face of $\Lambda_k$, for example $z_1 = k$. Let $\bar{z}$ be the reflection of $z$ in the first coordinate i.e. $\bar{z} = (-z_1, z_2, \ldots, z_d)$. By reflection symmetry we also have $\mathbb{P}(0 \xrightarrow{\Lambda_k} \bar{z}) \geq ck^{1-d}$. Translating $z$ to $x$ and $\bar{z}$ to $y$ gives

$$\mathbb{P}(x \xrightarrow{x+\Lambda_k} x + z), \mathbb{P}(y \xrightarrow{y+\Lambda_k} y + \bar{z}) \geq \frac{c}{k^{d-1}}.$$
Lemma (Cerf, 2015)

For any \( x, y \in \Lambda_n \), \( \mathbb{P}_{pc}(x \xleftrightarrow{\Lambda_{2n}} y) \geq cn^{-C} \).

Proof.

Assume first that \( x - y = (2k, 0, \ldots, 0) \), \( k \leq n \). By the theorem there exists a \( z \in \partial\Lambda_k \) such that

\[
\mathbb{P}(0 \xleftrightarrow{\Lambda_k} z) \geq \frac{1}{2d|\partial\Lambda_k|} \geq \frac{c}{k^{d-1}}.
\]

By rotation and reflection symmetry we may assume \( z \) is in some face of \( \Lambda_k \), for example \( z_1 = k \). Let \( \bar{z} \) be the reflection of \( z \) in the first coordinate i.e. \( \bar{z} = (-z_1, z_2, \ldots, z_d) \). By reflection symmetry we also have \( \mathbb{P}(0 \xleftrightarrow{\Lambda_k} \bar{z}) \geq ck^{1-d} \). Translating \( z \) to \( x \) and \( \bar{z} \) to \( y \) gives

\[
\mathbb{P}(x \xleftrightarrow{\Lambda_k} x + z), \mathbb{P}(y \xleftrightarrow{\Lambda_k} y + \bar{z}) \geq \frac{c}{k^{d-1}}.
\]

But \( x + z = y + \bar{z} \)!
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xrightarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. Then there exists a $z$ such that

$$\mathbb{P}(x \xrightarrow{x+\Lambda_{k}} x + z), \mathbb{P}(y \xrightarrow{y+\Lambda_{k}} x + z) \geq \frac{c}{k^{d-1}}.$$
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Since \( x + \Lambda_k \subset \Lambda_{2n} \) and ditto for \( y + \Lambda_k \) we can write

\[
\mathbb{P}(x \xleftarrow{\Lambda_{2n}} x + z), \mathbb{P}(y \xleftarrow{\Lambda_{2n}} x + z) \geq \frac{c}{k^{d-1}}.
\]
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \leftrightarrow_{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. Then there exists a $z$ such that

$$\mathbb{P}(x \leftrightarrow_{x+\Lambda_k} x + z), \mathbb{P}(y \leftrightarrow_{y+\Lambda_k} x + z) \geq \frac{c}{k^{d-1}}.$$ 

Since $x + \Lambda_k \subset \Lambda_{2n}$ and ditto for $y + \Lambda_k$ we can write

$$\mathbb{P}(x \leftrightarrow_{\Lambda_{2n}} x + z), \mathbb{P}(y \leftrightarrow_{\Lambda_{2n}} x + z) \geq \frac{c}{k^{d-1}}.$$ 

By FKG

$$\mathbb{P}(x \leftrightarrow_{\Lambda_{2n}} y) \geq \mathbb{P}(x \leftrightarrow_{\Lambda_{2n}} x + z, y \leftrightarrow_{\Lambda_{2n}} y + z) \geq \frac{c}{k^{2d-2}}.$$ 

Proving the lemma in this case.
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{p_c}(x \xrightarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. Then

$\mathbb{P}(x \xrightarrow{\Lambda_{2n}} y) \geq ck^{2-2d} \geq cn^{2-2d}$. 
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xleftarrow{\Lambda_{2n}} y) \geq cn^{-C}$.

Proof.

Assume first that $x - y = (2k, 0, \ldots, 0)$, $k \leq n$. Then
$\mathbb{P}(x \xleftarrow{\Lambda_{2n}} y) \geq ck^{2-2d} \geq cn^{2-2d}$. With a slightly smaller $c$, we can remove the requirement that the distance between $x$ and $y$ is even.
Lemma (Cerf, 2015)

For any \(x, y \in \Lambda_n\), \(\mathbb{P}_{pc}(x \xleftarrow{\Lambda_{2^n}} y) \geq cn^{-C}\).

Proof.

Assume first that \(x - y = (2k, 0, \ldots, 0), \ k \leq n\). Then
\[\mathbb{P}(x \xleftarrow{\Lambda_{2^n}} y) \geq ck^{2-2d} \geq cn^{2-2d}.\]
With a slightly smaller \(c\), we can remove the requirement that the distance between \(x\) and \(y\) is even. If they are not on a line, we define
\[x = x_0, \ldots, x_d = y\]
such that each couple \(x_i, x_{i+1}\) differ by only one coordinate.
Lemma (Cerf, 2015)

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Lemma (Cerf, 2015)

For any \( x, y \in \Lambda_n \), \( \mathbb{P}_{pc}(x \leftrightarrow^\Lambda_{2n} y) \geq cn^{-C} \).

Proof.

Assume first that \( x - y = (2k, 0, \ldots, 0), \ k \leq n \). Then
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\mathbb{P}(x \leftrightarrow^\Lambda_{2n} y) \geq ck^{2-2d} \geq cn^{2-2d}.
\]
With a slightly smaller \( c \), we can remove the requirement that the distance between \( x \) and \( y \) is even. If they are not on a line, we define
\[
x = x_0, \ldots, x_d = y
\]
such that each couple \( x_i, x_{i+1} \) differ by only one coordinate. Hence \( \mathbb{P}(x_i \leftrightarrow^\Lambda_{2n} x_{i+1}) \geq cn^{2-2d} \). Using FKG again gives
\[
\mathbb{P}(x \leftrightarrow^\Lambda_{2n} y) \geq \mathbb{P}(x_0 \leftrightarrow^\Lambda_{2n} x_1, x_1 \leftrightarrow^\Lambda_{2n} x_2, \ldots, x_{d-1} \leftrightarrow^\Lambda_{2n} x_d) \geq \prod_{i=1}^{d} \mathbb{P}(x_{i-1} \leftrightarrow^\Lambda_{2n} x_i) \geq \frac{c}{n^{2d^2-2d}}.
\]
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \leftrightarrow_{\Lambda^{2n}} y) \geq c n^{2d-2d^2}$. 
Lemma (Cerf, 2015)

For any $x, y \in \Lambda_n$, $\mathbb{P}_{pc}(x \xrightleftharpoons{\Lambda_{2n}} y) \geq cn^{2d-2d^2}$.

This was recently improved to $cn^{-d^2}$ by van den Berg and Don.
**Lemma (Cerf, 2015)**

\[
\text{For any } x, y \in \Lambda_n, \quad \mathbb{P}_{p_c}(x \xleftrightarrow{\Lambda_{2^n}} y) \geq cn^{2d-2d^2}.
\]

This was recently improved to \( cn^{-d^2} \) by van den Berg and Don. Their proof has an interesting topological component.
Crossing probabilities

Let $\Lambda$ be a box in $\mathbb{Z}^d$, with the side lengths not necessarily equal. A crossing is an open path from one side of the box to the other.

**Easy way**

**Hard way**
Crossing probabilities

Let $\Lambda$ be a box in $\mathbb{Z}^d$, with the side lengths not necessarily equal. A crossing is an open path from one side of the box to the other.

**Theorem**

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

$$\mathbb{P}_{p_c}(\Lambda \text{ has an easy-way crossing}) > c$$
Crossing probabilities

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**Proof (Kesten? Bollobás-Riordan? Nolin?)**
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It is easier to draw in $d = 2$ so let us do this.
Crossing probabilities

Let $\Lambda$ be a box in $\mathbb{Z}^d$, with the side lengths not necessarily equal. A crossing is an open path from one side of the box to the other.

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**Proof (Kesten? Bollobás-Riordan? Nolin?)**

It is easier to draw in $d = 2$ so let us do this. Let $p(a, b)$ be the probability of an easy-way crossing of an $a \times b$ rectangle.
Crossing probabilities

Let \( \Lambda \) be a box in \( \mathbb{Z}^d \), with the side lengths not necessarily equal. A crossing is an open path from one side of the box to the other.

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Let \( \Lambda \) be an \( 2n \times \cdots \times 2n \times n \) box in \( \mathbb{Z}^d \). Then

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**Proof (Kesten? Bollobás-Riordan? Nolin?)**

It is easier to draw in \( d = 2 \) so let us do this. Let \( p(a, b) \) be the probability of an easy-way crossing of an \( a \times b \) rectangle. We first claim that \( p(4n, n) \leq 5p(2n, n) \).
Crossing probabilities

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Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

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Theorem

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then $P_{pc}(\Lambda \text{ has an easy-way crossing}) > c$

Proof.

It is easier to draw in $d = 2$ so let us do this. Let $p(a, b)$ be the probability of an easy-way crossing of an $a \times b$ rectangle. We first claim that $p(4n, n) \leq 5p(2n, n)$. This is because if some path $\gamma$ crosses from the top to the bottom of a $4n \times n$ rectangle, it must cross either one of 3 horizontal rectangles or one of two vertical ones.
Crossing probabilities

**Theorem**

Let \( \Lambda \) be an \( 2n \times \cdots \times 2n \times n \) box in \( \mathbb{Z}^d \). Then

\[
P_{pc}(\Lambda \text{ has an easy-way crossing}) > c
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**Proof.**

It is easier to draw in \( d = 2 \) so let us do this. Let \( p(a, b) \) be the probability of an easy-way crossing of an \( a \times b \) rectangle. We first claim that \( p(4n, n) \leq 5p(2n, n) \). This is because if some path \( \gamma \) crosses from the top to the bottom of a \( 4n \times n \) rectangle, it must cross either one of 3 horizontal rectangles or one of two vertical ones. We next claim that \( p(4n, 2n) \leq p(4n, n)^2 \).
Crossing probabilities

**Theorem**

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

\[ P_{p_c} (\Lambda \text{ has an easy-way crossing}) > c \]

**Proof.**

It is easier to draw in $d = 2$ so let us do this. Let $p(a, b)$ be the probability of an easy-way crossing of an $a \times b$ rectangle. We first claim that $p(4n, n) \leq 5p(2n, n)$. This is because if some path $\gamma$ crosses from the top to the bottom of a $4n \times n$ rectangle, it must cross either one of 3 horizontal rectangles or one of two vertical ones. We next claim that $p(4n, 2n) \leq p(4n, n)^2$. But that means that $p(4n, 2n) \leq 25p(2n, n)^2$ and inductively that $p(2^{k+1}n, 2^kn) \leq 25^{2^{k}-1}p(2n, n)^{2^k}$. 
Crossing probabilities

**Theorem**

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

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It is easier to draw in $d = 2$ so let us do this. Let $p(a, b)$ be the probability of an easy-way crossing of an $a \times b$ rectangle. We first claim that $p(4n, n) \leq 5p(2n, n)$. This is because if some path $\gamma$ crosses from the top to the bottom of a $4n \times n$ rectangle, it must cross either one of 3 horizontal rectangles or one of two vertical ones. We next claim that $p(4n, 2n) \leq p(4n, n)^2$. But that means that $p(4n, 2n) \leq 25p(2n, n)^2$ and inductively that $p(2^{k+1}n, 2^kn) \leq 25^{2^k-1}p(2n, n)^{2^k}$. Thus, if for some $n$, $p(2n, n) < \frac{1}{25}$, then it decays exponentially, contradicting the result that $\chi(p_c) = \infty$. \qed
Theorem

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

$\mathbb{P}_{pc}(\Lambda \text{ has an easy-way crossing}) > c$

It is natural to ask if there is a corresponding upper bound, namely is it true that

$\mathbb{P}_{pc}(\Lambda \text{ has an easy-way crossing}) \leq 1 - c$

for some $c > 0$? This is true when $d = 2$. It is false for $d > 6$, in fact

$\mathbb{P}_{pc}(\Lambda \text{ has an easy-way crossing}) \to 1$ as $n \to \infty$.

It is not known in intermediate dimensions. In dimensions 2 and high, there is no significant difference between easy-way and hard-way crossing. In intermediate dimensions this is not known.
Crossing probabilities

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P_{pc}(\Lambda \text{ has an easy-way crossing}) \rightarrow 1 \text{ as } n \rightarrow \infty.
\]
Crossing probabilities

**Theorem**

Let $\Lambda$ be an $2n \times \cdots \times 2n \times n$ box in $\mathbb{Z}^d$. Then

$$\mathbb{P}_{p_c}(\Lambda \text{ has an easy-way crossing}) > c$$

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One arm exponent

**Theorem**

\[ P(0 \leftrightarrow \partial \Lambda_n) > c/n^{(d-1)/2}. \]
One arm exponent

**Theorem**

\[ \mathbb{P}(0 \leftrightarrow \partial \Lambda_n) > c/n^{(d-1)/2}. \]

**Proof.**

By the previous theorem we know that the box 
\[ [-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n] \] has an easy-way crossing with probability at least \( c \).
One arm exponent

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Proof.
By the previous theorem we know that the box 
\([-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n]\) has an easy-way crossing with probability at least \(c\). “Easy-way” means from 
\(\{n/2\} \times [-n, n]^{d-1}\) to \(\{-n/2\} \times [-n, n]^{d-1}\) so it must cross 
\(0 \times [-n, n]^{d-1}\).
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\(0 \times [-n, n]^{d-1}\). Therefore there exists some \(x \in \{0\} \times [-n, n]^{d-1}\) such that the probability that the crossing pass through it is at least \(c/n^{d-1}\).
One arm exponent

**Theorem**

$$\mathbb{P}(0 \leftrightarrow \partial \Lambda_n) > \frac{c}{n^{(d-1)/2}}.$$

**Proof.**

By the previous theorem we know that the box $$[-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n]$$ has an easy-way crossing with probability at least $$c$$. “Easy-way” means from $$\{n/2\} \times [-n, n]^{d-1}$$ to $$\{-n/2\} \times [-n, n]^{d-1}$$ so it must cross $$0 \times [-n, n]^{d-1}$$. Therefore there exists some $$x \in \{0\} \times [-n, n]^{d-1}$$ such that the probability that the crossing pass through it is at least $$c/n^{d-1}$$. But if it does, then $$x$$ is connected to distance at least $$n/2$$ by *two disjoint* paths.

The BK inequality finishes the proof.

In $$d = 2$$ Kesten improved this to $$n^{-1/3}$$.
One arm exponent

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\[ \mathbb{P}(0 \leftrightarrow \partial \Lambda_n) > c/n^{(d-1)/2}. \]

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By the previous theorem we know that the box \([-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n]\) has an easy-way crossing with probability at least \(c\). “Easy-way” means from \(\{n/2\} \times [-n, n]^{d-1}\) to \(\{-n/2\} \times [-n, n]^{d-1}\) so it must cross \(0 \times [-n, n]^{d-1}\). Therefore there exists some \(x \in \{0\} \times [-n, n]^{d-1}\) such that the probability that the crossing pass through it is at least \(c/n^{d-1}\). But if it does, then \(x\) is connected to distance at least \(n/2\) by *two disjoint* paths. The BK inequality finishes the proof.
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\[ \Pr(0 \leftrightarrow \partial \Lambda_n) > c/n^{(d-1)/2}. \]

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By the previous theorem we know that the box \([-n/2, n/2] \times [-n, n] \times \cdots \times [-n, n]\) has an easy-way crossing with probability at least \(c\). “Easy-way” means from \(\{n/2\} \times [-n, n]^{d-1}\) to \(\{-n/2\} \times [-n, n]^{d-1}\) so it must cross \(0 \times [-n, n]^{d-1}\). Therefore there exists some \(x \in \{0\} \times [-n, n]^{d-1}\) such that the probability that the crossing pass through it is at least \(c/n^{d-1}\). But if it does, then \(x\) is connected to distance at least \(n/2\) by *two disjoint* paths. The BK inequality finishes the proof.

In \(d = 2\) Kesten improved this to \(n^{-1/3}\).
\[ \chi(p_c) = \infty \]

\[ \sum_{x \in \partial \Lambda_n} \mathbb{P}_{p_c}(0 \xleftrightarrow{\Lambda_n} x) \geq 1 \]

\[ \mathbb{P}_{p_c}(x \xleftrightarrow{\Lambda_{2n}} y) > cn^{-C} \]

\[ \mathbb{P}_{p_c}(0 \leftrightarrow \partial \Lambda_n) > cn^{(1-d)/2} \]
The Aizenman-Kesten-Newman argument
Let $E$ be the number of open edges in $\mathcal{C}(0)$ and let $B$ be the number of closed edges in its boundary.
Lemma

Let $E$ be the number of open edges in $C(0)$ and let $B$ be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter.
Lemma

Let $E$ be the number of open edges in $C(0)$ and let $B$ be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter. Then

$$\mathbb{P}_p(B + E \leq n, |(1 - p)E - pB| > \lambda \sqrt{n}) \leq Ce^{-c\lambda^2}.$$
Lemma

Let $E$ be the number of open edges in $C(0)$ and let $B$ be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter. Then

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**Exploration and martingales**

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The lemma follows from Azuma-Hoeffding.
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This is a flexible argument. You can start from a set of vertices (not just one), and you can add additional stopping conditions.
Exploration and martingales

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Let $E$ be the number of open edges in $G(0)$ and let $B$ be the number of closed edges in its boundary. Let $\lambda > 0$ be some parameter. Then

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This is a flexible argument. You can start from a set of vertices (not just one), and you can add additional stopping conditions. For example,

**Lemma**

Let $S \subset \Lambda$ be the set of vertices connected to the boundary. Let $E$ be the number of open edges between vertices of $S$ and let $B$ be the number of closed edges with at least one vertex in $S$ and both vertices in $\Lambda$.  

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$$
\mathbb{P}(|X| > \lambda n^{d/2}) \leq e^{-c\lambda^2}.
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Notation

Let $A$, $B$ be subsets of $E \subseteq \mathbb{Z}^d$. We denote by

$$A \leftrightarrow_{E} B$$

the event that there are two disjoint clusters in $E$ which intersect both $A$ and $B$. 
Let $A, B$ be subsets of $E \subseteq \mathbb{Z}^d$. We denote by

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the event that there are two disjoint clusters in $E$ which intersect both $A$ and $B$. We will use very often $A \leftrightarrow^E \partial E$ and in this case we omit the superscript, i.e. write $A \leftrightarrow \partial E$. 
Theorem

Let $V$ be the number of edges $(x, y)$ in $\Lambda_n$ such that
\[
\{x, y\} \leftrightarrow \partial \Lambda_n \text{ i.e. both } x \text{ and } y \text{ are connected to } \partial \Lambda_n \text{ but } x \not\leftrightarrow y.
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$|X(\bigcup_i \mathcal{C}_i)| < Cn^{d/2}\sqrt{\log n}, |X(\mathcal{C}_i)| < C\sqrt{|\mathcal{C}_i|}\sqrt{\log n}$ for all $i$. 

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“with high probability” can be made to mean “with probability \( > 1 - n^{-1/2} \)” and we are done.
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Corollary

For $x$ a neighbour of 0,

$$\mathbb{P} (\{0, x\} \leftrightarrow \partial \Lambda_n) < C \sqrt{\frac{\log n}{n}}.$$
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$$X(L \cup R) - X(L) - X(R)$$

teaches something about edges connected to both the left and the right.
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teaches something about edges connected to both the left and the right. Hutchcroft has a version where one explores from random points.
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]
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Recall from the previous slide

Corollary

For \( x \) a neighbour of 0, \( P(\{0, x\} \leftrightarrow \partial \Lambda_n) < C\sqrt{\log n}/n \).
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$\mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq C n^{-c}$ for $c > 0$ small enough.

Let $k < \frac{1}{2} n$ be some number (it will be $n^c$ eventually, but for now let us keep it a parameter).
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**Proof.**

Let \( x \in A \cap \Lambda_k \) and \( y \in B \cap \Lambda_k \).
Theorem (Cerf, 2015)

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Proof.

Let \( x \in A \cap \Lambda_k \) and \( y \in B \cap \Lambda_k \). With probability at least \( ck^{2d-2d^2} \) there is an open path \( \gamma \) from \( x \) to \( y \).
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**Lemma**

Let $A, B \subset \Lambda_{2k}$, both intersecting $\Lambda_k$. Then

$$\mathbb{P}_{p_c}(A \xleftarrow{\Lambda_{2k} \setminus A \cup B} \rightarrow B) > ck^{2d-2d^2}.$$

**Proof.**

Let $x \in A \cap \Lambda_k$ and $y \in B \cap \Lambda_k$. With probability at least $ck^{2d-2d^2}$ there is an open path $\gamma$ from $x$ to $y$. The portion of $\gamma$ from its last vertex in $A$ until the first vertex in $B$ after it demonstrates the lemma.
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^c} \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xrightarrow{\Lambda_{2k} \setminus A \cup B} B) > ck^{2d-2d^2}. \]
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^c} \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \overset{\Lambda_{2k} \setminus A \cup B}{\leftrightarrow} B) > ck^{2d-2d^2}. \]

Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xrightarrow{\Lambda_{2k} \setminus A \cup B} B) > ck^{2d-2d^2}. \]

Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).

Lemma

\textit{Let } \( E \text{ be the event that there exist edges } e, f \in \Lambda_{2k} \text{ such that } \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \text{ but } e^- \leftrightarrow e^+, f^- \leftrightarrow f^+ \text{ and } e^- \leftrightarrow f^+ \).
Lemma

Let $E$ be the event that there exist edges $e, f \in \Lambda_{2k}$ such that $\partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n$ but $e^- \leftrightarrow e^+$, $f^- \leftrightarrow f^+$ and $e^- \leftrightarrow f^+$. Then $P(E) \geq c k^{-2} d^2 P$. 
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \leftarrow_{\Lambda_{2k}} A \cup B \rightarrow B) > ck^{2d-2d^2}. \]
Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).

Lemma

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that
\( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+ \), \( f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P. \)
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n c \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xleftarrow{\Lambda_{2k} \setminus A \cup B} B) > c k^{2d-2d^2}. \]

Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).

Lemma

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that

\( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+ \), \( f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq c k^{-2d^2} P \).

Proof.

There exist some \( x, y \in \Lambda_k \) such that with probability \( k^{-2d} P \),
\( x \leftrightarrow \partial \Lambda_n \), \( y \leftrightarrow \partial \Lambda_n \) and \( x \leftrightarrow y \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^c} \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xrightarrow{\Lambda_{2k} \setminus A \cup B} B) > ck^{2d-2d^2}. \]

Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).

Lemma

Let \( E \) be the event that there exist edges \( e,f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+ \), \( f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2}P \).

Proof.

There exist some \( x,y \in \Lambda_k \) such that with probability \( k^{-2d}P \), \( x \leftrightarrow \partial \Lambda_n \), \( y \leftrightarrow \partial \Lambda_n \) and \( x \leftrightarrow y \). Condition on \( \mathcal{C}(x) \) and \( \mathcal{C}(y) \).
**Theorem (Cerf, 2015)**

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

**Lemma:** Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xleftarrow{\Lambda_{2k} \setminus A \cup B} \, B) > ck^{2d-2d^2}. \]

Denote \( P := \mathbb{P}({\Lambda_k \leftrightarrow \partial \Lambda_n}) \).

**Lemma**

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that

\[ \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \] but \( e^- \leftrightarrow e^+, f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P \).

**Proof.**

There exist some \( x, y \in \Lambda_k \) such that with probability \( k^{-2d} P \), \( x \leftrightarrow \partial \Lambda_n, y \leftrightarrow \partial \Lambda_n \) and \( x \leftrightarrow y \). Condition on \( \mathcal{C}(x) \) and \( \mathcal{C}(y) \). Use the previous lemma with \( A = \overline{\mathcal{C}(x)} \) i.e. \( \mathcal{C}(x) \) with its immediate neighbourhood and \( B = \overline{\mathcal{C}(y)} \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma: Let \( A, B \subset \Lambda_{2k} \), both intersecting \( \Lambda_k \). Then

\[ \mathbb{P}_{p_c}(A \xrightarrow{\Lambda_{2k} \setminus A \cup B} B) > ck^{2d-2d^2}. \]

Denote \( P := \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_n) \).

Lemma

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+ \), \( f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2}P \).

Proof.

There exist some \( x, y \in \Lambda_k \) such that with probability \( k^{-2d}P \), \( x \leftrightarrow \partial \Lambda_n, y \leftrightarrow \partial \Lambda_n \) and \( x \leftrightarrow y \). Condition on \( \mathcal{C}(x) \) and \( \mathcal{C}(y) \).

Use the previous lemma with \( A = \overline{\mathcal{C}(x)} \) i.e. \( \mathcal{C}(x) \) with its immediate neighbourhood and \( B = \overline{\mathcal{C}(y)} \). \( A \xrightarrow{\Lambda_{2k} \setminus A \cup B} B \) is independent of the conditioning.
**Lemma**

Let $E$ be the event that there exist edges $e, f \in \Lambda_{2k}$ such that
$\partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n$ but $e^- \leftrightarrow e^+$, $f^- \leftrightarrow f^+$ and $e^- \leftrightarrow f^+$. Then $\mathbb{P}_{pc}(E) \geq ck^{-2d^2} P$.

**Proof.**

There exist some $x, y \in \Lambda_k$ such that with probability $k^{-2d} P$, $x \leftrightarrow \partial \Lambda_n$, $y \leftrightarrow \partial \Lambda_n$ and $x \leftrightarrow y$. Condition on $C(x)$ and $C(y)$. Use the previous lemma with $A = \overline{C(x)}$ i.e. $C(x)$ with its immediate neighbourhood and $B = \overline{C(y)}$. $A \xleftarrow{\Lambda_{2k} \setminus A \cup B} B$ is independent of the conditioning.
**Lemma**

Let $E$ be the event that there exist edges $e,f \in \Lambda_{2k}$ such that
\[
\partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \text{ but } e^- \leftrightarrow e^+, \ f^- \leftrightarrow f^+ \text{ and } e^- \leftrightarrow f^+.
\]
Then $\mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P$.

**Proof.**

There exist some $x, y \in \Lambda_k$ such that with probability $k^{-2d} P$, $x \leftrightarrow \partial \Lambda_n$, $y \leftrightarrow \partial \Lambda_n$ and $x \leftrightarrow y$. Condition on $C(x)$ and $C(y)$. Use the previous lemma with $A = \overline{C(x)}$ i.e. $C(x)$ with its immediate neighbourhood and $B = \overline{C(y)}$. $A \xrightarrow{A \cup B} B$ is independent of the conditioning.

This kind of argument is called a “patching argument”.

**Theorem (Cerf, 2015)**

\[ \mathbb{P}_{p_c}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

**Lemma**

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+ \), \( f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+, f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P \).

Proof of the theorem.

For given edges \( e \) and \( f \) denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)).
**Theorem (Cerf, 2015)**

\[ \mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

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**Lemma**

*Let E be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \leftrightarrow e^+, f^- \leftrightarrow f^+ \) and \( e^- \leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq ck^{-2d^2} P. \)*

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**Proof of the theorem.**

For given edges \( e \) and \( f \) denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)). Choose some \( e \) and \( f \) such that \( \mathbb{P}(E_{e,f}) \geq ck^{-2d^2-2d} P. \)
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

Lemma

Let \( E \) be the event that there exist edges \( e, f \in \Lambda_{2k} \) such that \( \partial \Lambda_n \leftrightarrow e^-, e^+ \leftrightarrow f^-, f^+ \leftrightarrow \partial \Lambda_n \) but \( e^- \not\leftrightarrow e^+ \), \( f^- \not\leftrightarrow f^+ \) and \( e^- \not\leftrightarrow f^+ \). Then \( \mathbb{P}_{p_c}(E) \geq c k^{-2d^2} P \).

Proof of the theorem.

For given edges \( e \) and \( f \) denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)). Choose some \( e \) and \( f \) such that \( \mathbb{P}(E_{e,f}) \geq c k^{-2d^2 - 2d} P \). The event \( E_{e,f}^* \) that “\( E_{e,f} \) would have been satisfied had \( e \) been closed, but it’s open” satisfies \( \mathbb{P}(E_{e,f}^*) \approx \mathbb{P}(E_{e,f}) \).
**Theorem (Cerf, 2015)**

\[ \mathbb{P}_{p_c}(\Lambda_n^c \iff \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

**Lemma**

*Let E be the event that there exist edges e, f \( \in \Lambda_{2k} \) such that \( \partial \Lambda_n \iff e^-, e^+ \iff f^-, f^+ \iff \partial \Lambda_n \) but \( e^- \iff e^+ \), \( f^- \iff f^+ \) and \( e^- \iff f^+ \). Then \( \mathbb{P}_{p_c} (E) \geq ck^{-2d^2} P \).*

**Proof of the theorem.**

For given edges e and f denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)). Choose some e and f such that

\[ \mathbb{P}(E_{e,f}) \geq ck^{-2d^2-2d} P. \]

The event \( E^*_e,f \) that “\( E_{e,f} \) would have been satisfied had e been closed, but it’s open” satisfies

\[ \mathbb{P}(E^*_e,f) \approx \mathbb{P}(E_{e,f}). \]

But \( E^*_e,f \) implies \( f \iff \partial \Lambda_n \), which has probability \( \leq C \sqrt{(\log n)/n} \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq C n^{-c} \text{ for } c > 0 \text{ small enough.} \]

Proof of the theorem.

For given edges \( e \) and \( f \) denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)). Choose some \( e \) and \( f \) such that \( \mathbb{P}(E_{e,f}) \geq c k^{-2d^2 - 2d} P \). The event \( E^*_{e,f} \) that “\( E_{e,f} \) would have been satisfied had \( e \) been closed, but it’s open” satisfies \( \mathbb{P}(E^*_{e,f}) \approx \mathbb{P}(E_{e,f}) \). But \( E^*_{e,f} \) implies \( f \leftrightarrow \partial \Lambda_n \), which has probability \( \leq C \sqrt{\frac{(\log n)}{n}} \). All in all we get

\[ C \sqrt{\frac{\log n}{n}} \geq \mathbb{P}(E^*_{e,f}) \geq c \mathbb{P}(E_{e,f}) \geq c k^{-2d^2 - 2d} P. \]
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq Cn^{-c} \text{ for } c > 0 \text{ small enough.} \]

Proof of the theorem.

For given edges \( e \) and \( f \) denote by \( E_{e,f} \) the event as in the lemma (so \( E = \bigcup E_{e,f} \)). Choose some \( e \) and \( f \) such that \( \mathbb{P}(E_{e,f}) \geq ck^{-2d^2-2d}P \). The event \( E^*_e,f \) that “\( E_{e,f} \) would have been satisfied had \( e \) been closed, but it’s open” satisfies \( \mathbb{P}(E^*_e,f) \approx \mathbb{P}(E_{e,f}) \). But \( E^*_e,f \) implies \( f \leftrightarrow \partial \Lambda_n \), which has probability \( \leq C\sqrt{\log n}/n \). All in all we get

\[
C\sqrt{\frac{\log n}{n}} \geq \mathbb{P}(E^*_e,f) \geq c\mathbb{P}(E_{e,f}) \geq ck^{-2d^2-2d}P.
\]

Choosing \( k = n^{1/(8d^2+8d)} \) proves the theorem.
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c} \left( \Lambda_{n^{1/(8d^2+8d)-o(1)}} \right) \iff \partial \Lambda_n \leq C n^{-1/4}. \]
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^{1/(8d^2+8d)-o(1)}} \iff \partial \Lambda_n) \leq Cn^{-1/4}. \]

- The theorem actually holds for all \( p \).
Theorem (Cerf, 2015)
\[ \mathbb{P}_{p_{c}}(\Lambda_{n^{1/(8d^2+8d)-o(1)}} \leftrightarrow \partial \Lambda_n) \leq C n^{-1/4} \]

- The theorem actually holds for all \( p \).
- Cerf had the scheme for improving the exponents.

Get a better estimate for \( \mathbb{P}(\Lambda_{n^c} \leftrightarrow \partial \Lambda_n) \)

Get a better estimate for the number of clusters from \( \partial \Lambda_{2n} \) to \( \partial \Lambda_n \)

\[ \sum \sqrt{|C|} \]

Get a better estimate for \( \mathbb{P}([0, x] \leftrightarrow \partial \Lambda) \)
The theorem actually holds for all $p$.

Cerf had the scheme for improving the exponents. Unfortunately, the end result was

$$\mathbb{P}(\{0, x\} \iff \partial \Lambda_n) \leq n^{-\frac{2d^2+3d-3}{4d^2+5d-5}+o(1)}$$

which is not a big improvement over $\frac{1}{2}$, say in $d = 3$ it gives $\frac{12}{23}$. 

$$\mathbb{P}_{pc}(\Lambda_{n^{1/(8d^2+8d)}-o(1)} \iff \partial \Lambda_n) \leq Cn^{-1/4}.$$
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{\frac{1}{(8d^2+8d)-o(1)}} \iff \partial \Lambda_n) \leq C n^{-1/4}. \]

- The theorem actually holds for all \( p \).
- Cerf had the a scheme for improving the exponents. Unfortunately, the end result was

\[ \mathbb{P}(\{0, x\} \iff \partial \Lambda_n) \leq n^{-\frac{2d^2+3d-3}{4d^2+5d-5}+o(1)} \]

which is not a big improvement over \( \frac{1}{2} \), say in \( d = 3 \) it gives \( \frac{12}{23} \).

Definition

Let \( \eta \) be some positive number smaller than \( \frac{1}{8d^2+8d} \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^{1/(8d^2+8d) - o(1)}} \Leftrightarrow \partial \Lambda_n) \leq C n^{-1/4}. \]

Lemma

Call a cluster \( C \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^{1/(8d^2+8d)-o(1)} \iff \partial \Lambda_n) \leq Cn^{-1/4}. \]

Lemma

Call a cluster \( \mathcal{C} \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \). Then

\[ \mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c. \]
**Theorem (Cerf, 2015)**

\[
\mathbb{P}_{p_c} \left( \Lambda_n^{1/(8d^2+8d)-o(1)} \Leftrightarrow \partial \Lambda_n \right) \leq Cn^{-1/4}.
\]

**Lemma**

Call a cluster \( \mathcal{C} \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \). Then

\[
\mathbb{P}_{p_c} (\exists \text{ large cluster}) \leq 1 - c.
\]

**Proof.**

Denote the event by \( E \). Assume both \( E \) and its translation by \((n/2, 0, \ldots, 0)\) occurred (call the translates \( \Lambda', \mathcal{C}' \) and \( E' \)).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_n^{1/(8d^2+8d)-o(1)} \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/4}. \]

Lemma

Call a cluster \( \mathcal{C} \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \). Then

\[ \mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c. \]

Proof.

Denote the event by \( E \). Assume both \( E \) and its translation by \((n/2, 0, \ldots, 0)\) occurred (call the translates \( \Lambda', \mathcal{C}' \) and \( E' \)). Then there at least \( \frac{1}{4} \) of the \( n^\eta \) cubes in \( \Lambda \cap \Lambda' \) intersect both \( \mathcal{C} \) and \( \mathcal{C}' \).
Theorem (Cerf, 2015)

\[ \mathbb{P}_{pc}(\Lambda_n^{1/(8d^2+8d) - o(1)} \Leftrightarrow \partial \Lambda_n) \leq Cn^{-1/4}. \]

Lemma

Call a cluster \( \mathcal{C} \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^n \) in \( \Lambda_n \). Then

\[ \mathbb{P}_{pc}(\exists \text{ large cluster}) \leq 1 - c. \]

Proof.

Denote the event by \( E \). Assume both \( E \) and its translation by \( (n/2, 0, \ldots, 0) \) occurred (call the translates \( \Lambda', \mathcal{C}' \) and \( E' \)). Then there at least \( \frac{1}{4} \) of the \( n^n \) cubes in \( \Lambda \cap \Lambda' \) intersect both \( \mathcal{C} \) and \( \mathcal{C}' \). If \( \mathcal{C} \neq \mathcal{C}' \) then each of these cubes satisfies the two disjoint clusters event.
Theorem (Cerf, 2015)

\[ \mathbb{P}_{p_c}(\Lambda_{n^{1/(8d^2+8d)-o(1)}^{\partial}}) \leq Cn^{-1/4}. \]

Lemma

Call a cluster \( C \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \). Then

\[ \mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c. \]

Proof.

Denote the event by \( E \). Assume both \( E \) and its translation by \((n/2, 0, \ldots, 0)\) occurred (call the translates \( \Lambda', C' \) and \( E' \)). Then there at least \( \frac{1}{4} \) of the \( n^\eta \) cubes in \( \Lambda \cap \Lambda' \) intersect both \( C \) and \( C' \). If \( C \neq C' \) then each of these cubes satisfies the two disjoint clusters event. Hence by Cerf’s theorem and Markov’s inequality

\[ \mathbb{P}_{p_c}(E \cap E' \cap \{C \neq C'\}) \leq Cn^{-1/4}. \]
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_n$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^n$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c_1$.

Proof.

Denote the event by $E$. Assume both $E$ and its translation by $(n/2, 0, \ldots, 0)$ occurred (call the translates $\Lambda', \mathcal{C}'$ and $E'$). Then there at least $\frac{1}{4}$ of the $n^n$ cubes in $\Lambda \cap \Lambda'$ intersect both $\mathcal{C}$ and $\mathcal{C}'$. If $\mathcal{C} \neq \mathcal{C}'$ then each of these cubes satisfies the two disjoint clusters event. Hence by Cerf’s theorem and Markov’s inequality $\mathbb{P}_{p_c}(E \cap E' \cap \{\mathcal{C} \neq \mathcal{C}'\}) \leq Cn^{-1/4}$. 

By continuity, the same inequality will hold for a slightly smaller $p$.
Lemma

Call a cluster \( C \) in \( \Lambda_n \) “large” if it intersects \( \frac{7}{8} \) of the cubes of side-length \( n^\eta \) in \( \Lambda_n \). Then \( \mathbb{P}_{p_c} (\exists \text{ large cluster}) \leq 1 - c_1 \).

Proof.

Denote the event by \( E \). Assume both \( E \) and its translation by \((n/2, 0, \ldots, 0)\) occurred (call the translates \( \Lambda', C' \) and \( E' \)). Then there at least \( \frac{1}{4} \) of the \( n^\eta \) cubes in \( \Lambda \cap \Lambda' \) intersect both \( C \) and \( C' \). If \( C \neq C' \) then each of these cubes satisfies the two disjoint clusters event. Hence by Cerf’s theorem and Markov’s inequality \( \mathbb{P}_{p_c} (E \cap E' \cap \{C \neq C'\}) \leq Cn^{-1/4} \). Hence

\[
\mathbb{P}_{p_c} (C = C') \geq 1 - 2c_1 - Cn^{-1/4}.
\]
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_n$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^n$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c_1$.

Proof.

Denote the event by $E$. Assume both $E$ and its translation by $(n/2, 0, \ldots, 0)$ occurred (call the translates $\Lambda'$, $\mathcal{C}'$ and $E'$). Then there at least $\frac{1}{4}$ of the $n^n$ cubes in $\Lambda \cap \Lambda'$ intersect both $\mathcal{C}$ and $\mathcal{C}'$. If $\mathcal{C} \neq \mathcal{C}'$ then each of these cubes satisfies the two disjoint clusters event. Hence by Cerf’s theorem and Markov’s inequality $\mathbb{P}_{p_c}(E \cap E' \cap \{\mathcal{C} \neq \mathcal{C}'\}) \leq Cn^{-1/4}$. Hence

$$\mathbb{P}_{p_c}(\mathcal{C} = \mathcal{C}') \geq 1 - 2c_1 - Cn^{-1/4}.$$ 

By continuity, the same inequality will hold for a slightly smaller $p$. 

Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_n$ "large" if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c_1$.

Proof.

Denote the event by $E$. Assume both $E$ and its translation by $(n/2, 0, \ldots, 0)$ occurred (call the translates $\Lambda', \mathcal{C}'$ and $E'$). Then there at least $\frac{1}{4}$ of the $n^\eta$ cubes in $\Lambda \cap \Lambda'$ intersect both $\mathcal{C}$ and $\mathcal{C}'$. If $\mathcal{C} \neq \mathcal{C}'$ then each of these cubes satisfies the two disjoint clusters event. Hence by Cerf’s theorem and Markov’s inequality $\mathbb{P}_{p_c}(E \cap E' \cap \{\mathcal{C} \neq \mathcal{C}'\}) \leq Cn^{-1/4}$. Hence

$$\mathbb{P}_{p_c}(\mathcal{C} = \mathcal{C}') \geq 1 - 2c_1 - Cn^{-1/4}.$$ 

By continuity, the same inequality will hold for a slightly smaller $p$. By a theorem of Liggett, Schonmann and Stacey (1997), if $c_1$ is sufficiently small and $n$ sufficiently large, then an infinite cluster exists, contradicting $p < p_c$. \qed
Lemma

Call a cluster $C$ in $\Lambda_n$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^n$ in $\Lambda_n$. Then $P_{pc}(\exists \text{ large cluster}) \leq 1 - c$.

The same argument works for clusters $\Lambda_{2n}$ (or any constant), i.e. we define the cluster by connections in $\Lambda_{2n}$ but still ask only about intersections with subcubes of $\Lambda_n$. 
Lemma

Call a cluster $C$ in $\Lambda_n$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists$ large cluster) $\leq 1 - c$.

The same argument works for clusters $\Lambda_{2n}$ (or any constant), i.e. we define the cluster by connections in $\Lambda_{2n}$ but still ask only about intersections with subcubes of $\Lambda_n$. The proof is the same, only the “distance of independence” in Liggett-Schonmann-Stacey needs to be increased.
Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c$.

The same argument works for clusters $\Lambda_{2n}$ (or any constant), i.e. we define the cluster by connections in $\Lambda_{2n}$ but still ask only about intersections with subcubes of $\Lambda_n$. The proof is the same, only the “distance of independence” in Liggett-Schonmann-Stacey needs to be increased.
Lemma

Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists$ large cluster) $\leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$. 
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists$ large cluster) $\leq 1 - c$.

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Proof.

Examine one $\nu$ (whose value will be chosen later) and assume by contradiction that this probability is, in fact, larger than $1 - Cn^{-d}$. 
Lemma

Call a cluster $C$ in $\Lambda_{2n}$ "large" if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{pc}(\exists$ large cluster) $\leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Examine one $\nu$ (whose value will be chosen later) and assume by contradiction that this probability is, in fact, larger than $1 - Cn^{-d}$. Then, with probability $> 1 - n^{-d\nu}$, each box $a + \Lambda_{n^\nu}$, $a \in \Lambda_n$ is connected to $a + \partial \Lambda_n$. 
Lemma

**Call a cluster 𝒞 in** \( \Lambda_{2n} \) **“large” if it** intersects \( \frac{7}{8} \) **of the cubes of side-length** \( n^n \) **in** \( \Lambda_n \). **Then** \( \mathbb{P}_{p_c} (\exists \text{ large cluster}) \leq 1 - c \).

Theorem (Duminil-Copin-K-Tassion, unpublished)

**For** \( d \geq 3 \) **and some** \( \nu = \nu(d) > 0 \), \( \mathbb{P}_{p_c} (\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > C n^{-d} \).

Proof.

Examine one \( \nu \) (whose value will be chosen later) and assume by contradiction that this probability is, in fact, larger than \( 1 - C n^{-d} \). Then, with probability \( > 1 - n^{-d \nu} \), **each box** \( a + \Lambda_{n^\nu}, a \in \Lambda_n \) **is connected to** \( a + \partial \Lambda_n \). **Denote this event by** \( A \).
Lemma

Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{pc}(\exists \text{ large cluster}) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Examine one $\nu$ (whose value will be chosen later) and assume by contradiction that this probability is, in fact, larger than $1 - Cn^{-d}$. Then, with probability $> 1 - n^{-d\nu}$, each box $a + \Lambda_{n^\nu}$, $a \in \Lambda_n$ is connected to $a + \partial \Lambda_n$. Denote this event by $A$. In particular, all boxes in $\Lambda_{n/4}$ are connected to $\partial \Lambda_{n/2}$. 
Lemma

Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{pc}(\exists \text{ large cluster}) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Examine one $\nu$ (whose value will be chosen later) and assume by contradiction that this probability is, in fact, larger than $1 - Cn^{-d}$. Then, with probability $> 1 - n^{-d\nu}$, each box $a + \Lambda_n^\nu$, $a \in \Lambda_n$ is connected to $a + \partial \Lambda_n$. Denote this event by $A$. In particular, all boxes in $\Lambda_{n/4}$ are connected to $\partial \Lambda_{n/2}$.

During this proof, whenever we say “cluster” we mean a cluster in $\Lambda_n$ that intersects $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$.
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > C n^{-d}$.

Proof.

$A \implies \{\text{all } n^\nu \text{ boxes in } \Lambda_{n/4} \text{ are connected to } \partial \Lambda_{n/2}\}$. 

Lemma

Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists$ large cluster $) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

$A \implies \{\text{all } n^\nu \text{ boxes in } \Lambda_{n/4} \text{ are connected to } \partial \Lambda_{n/2}\}$. For every cluster $C$ let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. 
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^n$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists$ large cluster) $\leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > C n^{-d}$.

Proof.

$A \implies \{\text{all } n^\nu \text{ boxes in } \Lambda_{n/4} \text{ are connected to } \partial \Lambda_{n/2}\}$. For every cluster $\mathcal{C}$ let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. Under $A$ we have,

$$n^{(1-\nu)d} \lesssim \sum_{\mathcal{C}} N(\mathcal{C})$$
Lemma

Call a cluster $C$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{p_c}(\exists \text{ large cluster}) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

$A \implies \{\text{all } n^\nu \text{ boxes in } \Lambda_{n/4} \text{ are connected to } \partial \Lambda_{n/2}\}$. For every cluster $C$ let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. Under $A$ we have, by concavity

$$n^{(1-\nu)d} \lesssim \sum_C N(C) \leq \left( \sum_C N(C)^{(d-1)/d} \right)^{d/(d-1)}.$$
Lemma

Call a cluster $\mathcal{C}$ in $\Lambda_{2n}$ “large” if it intersects $\frac{7}{8}$ of the cubes of side-length $n^\eta$ in $\Lambda_n$. Then $\mathbb{P}_{pc}(\exists \text{ large cluster}) \leq 1 - c$.

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

$A \implies \{\text{all } n^\nu \text{ boxes in } \Lambda_{n/4} \text{ are connected to } \partial \Lambda_{n/2}\}$. For every cluster $\mathcal{C}$ let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. Under $A$ we have, by concavity

$$n^{(1-\nu)d} \lesssim \sum_{\mathcal{C}} N(\mathcal{C}) \leq \left(\sum_{\mathcal{C}} N(\mathcal{C})^{(d-1)/d}\right)^{d/(d-1)}.$$

By the lemma,

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For \( d \geq 3 \) and some \( \nu = \nu(d) > 0 \), \( \mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d} \).

Proof.

Let \( N(\mathcal{C}) \) be the number of \( n^\nu \)-subboxes of \( \Lambda_n/2 \) that intersect \( \mathcal{C} \). \[ \mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)} \].
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda^n_{\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C}_{\text{small}}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$. Let us return to the proof of Gandolfi-Grimmett-Russo. In fact it shows that

$$\mathbb{P}(0 \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2} + C(\log n)n^{-d} \mathbb{E} \sum_{\mathcal{C}} \sqrt{|\mathcal{C}|}.$$ 

where the sum is over clusters in $\Lambda_{n/2}$. 

Theorem (Duminil-Copin-K-Tassion, unpublished)

For \( d \geq 3 \) and some \( \nu = \nu(d) > 0 \), \( \mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d} \).

Proof.

Let \( N(\mathcal{C}) \) be the number of \( n^\nu \)-subboxes of \( \Lambda_{n/2} \) that intersect \( \mathcal{C} \). \( \mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)} \). Let us return to the proof of Gandolfi-Grimmett-Russo. In fact it shows that

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\]

where the sum is over clusters in \( \Lambda_{n/2} \). A variation on the argument, also due to Cerf, shows that one can take the sum only over \( \mathcal{C} \) in \( \Lambda_n \) that intersect \( \Lambda_{n/4} \) and \( \partial \Lambda_{n/2} \).
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n\leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E}\sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$. Let us return to the proof of Gandolfi-Grimmett-Russo. In fact it shows that

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where the sum is over clusters in $\Lambda_{n/2}$. A variation on the argument, also due to Cerf, shows that one can take the sum only over $\mathcal{C}$ in $\Lambda_n$ that intersect $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$. And with Cerf’s lemma,

$$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{C\nu} \mathbb{P}(0 \leftrightarrow \partial \Lambda_{n/4})$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$. Let us return to the proof of Gandolfi-Grimmett-Russo. In fact it shows that

$$\mathbb{P}(0 \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2} + C(\log n)n^{-d} \mathbb{E} \sum_{\mathcal{C}} \sqrt{|\mathcal{C}|}.$$  

where the sum is over clusters in $\Lambda_{n/2}$. A variation on the argument, also due to Cerf, shows that one can take the sum only over $\mathcal{C}$ in $\Lambda_n$ that intersect $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$. And with Cerf’s lemma,

$$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{C\nu} \mathbb{P}(0 \leftrightarrow \partial \Lambda_{n/4})$$

$$\leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. $\mathbb{E} \sum_{C \text{ small}} N(C)^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{C} \sqrt{N(C)}$. 
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$. The isoperimetric inequality in $\mathbb{Z}^d$ shows that for every small $\mathcal{C}$ we have at least $cN(\mathcal{C})^{(d-1)/d}$ subboxes of $\Lambda_{n/2}$ which intersect $\mathcal{C}$ but have a neighbouring box that does not intersect $\mathcal{C}$. 

Theorem (Duminil-Copin-K-Tassion, unpublished)

For \(d \geq 3\) and some \(\nu = \nu(d) > 0\), \(\mathbb{P}_{p_c}(\Lambda_n^{\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}\).

Proof.

Let \(N(\mathcal{C})\) be the number of \(n^\nu\)-subboxes of \(\Lambda_{n/2}\) that intersect \(\mathcal{C}\). \(\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}\).
\(\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}\). The isoperimetric inequality in \(\mathbb{Z}^d\) shows that for every small \(\mathcal{C}\) we have at least \(cN(\mathcal{C})^{(d-1)/d}\) subboxes of \(\Lambda_{n/2}\) which intersect \(\mathcal{C}\) but have a neighbouring box that does not intersect \(\mathcal{C}\). Let \(Q\) be such a box and let \(Q'\) be its neighbour that does not intersect \(\mathcal{C}\).
For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cN^{(1-\nu)(d-1)}$. $\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$. The isoperimetric inequality in $\mathbb{Z}^d$ shows that for every small $\mathcal{C}$ we have at least $cN(\mathcal{C})^{(d-1)/d}$ subboxes of $\Lambda_{n/2}$ which intersect $\mathcal{C}$ but have a neighbouring box that does not intersect $\mathcal{C}$. Let $Q$ be such a box and let $Q'$ be its neighbour that does not intersect $\mathcal{C}$. Under the event $A$ (which, recall, said that every $n^\nu$ subbox of $\Lambda_{n/2}$ is connected to distance $n$), this implies that $Q \cup Q'$ is connected to distance $n/4$ by two disjoint clusters.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$. The isoperimetric inequality in $\mathbb{Z}^d$ shows that for every small $\mathcal{C}$ we have at least $cN(\mathcal{C})^{(d-1)/d}$ subboxes of $\Lambda_{n/2}$ which intersect $\mathcal{C}$ but have a neighbouring box that does not intersect $\mathcal{C}$. Let $Q$ be such a box and let $Q'$ be its neighbour that does not intersect $\mathcal{C}$. Under the event $A$ (which, recall, said that every $n^\nu$ subbox of $\Lambda_{n/2}$ is connected to distance $n$), this implies that $Q \cup Q'$ is connected to distance $n/4$ by two disjoint clusters. Thus, under $A$, there are $c \sum_{\mathcal{C}} N(\mathcal{C})^{(d-1)/d}$ boxes of size $2n^\nu$ in $\Lambda_{n/2}$ which are connected to distance $n/4$ by two disjoint clusters.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$.

Thus, under $A$, there are $c \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}$ boxes of size $2n^\nu$ in $\Lambda_{n/2}$ which are connected to distance $n/4$ by two disjoint clusters.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

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Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$.

Thus, under $A$, there are $c \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}$ boxes of size $2n^\nu$ in $\Lambda_{n/2}$ which are connected to distance $n/4$ by two disjoint clusters. There is some over-counting in this argument, every $2n^\nu$ box might be counted for every cluster that intersects it.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$.

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}.$$

$$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})}.$$

Thus, under $A$, there are $c \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}$ boxes of size $2n^\nu$ in $\Lambda_{n/2}$ which are connected to distance $n/4$ by two disjoint clusters. There is some over-counting in this argument, every $2n^\nu$ box might be counted for every cluster that intersects it. We bound the over-counting crudely by the volume of the box, $Cn^{d\nu}$. 

Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{pc}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. $\mathbb{E} \sum_{C \text{ small}} N(C)^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{C} \sqrt{N(C)}$.

Thus, under $A$, there are $c \sum_{C \text{ small}} N(C)^{(d-1)/d}$ boxes of size $2n^\nu$ in $\Lambda_{n/2}$ which are connected to distance $n/4$ by two disjoint clusters. There is some over-counting in this argument, every $2n^\nu$ box might be counted for every cluster that intersects it. We bound the over-counting crudely by the volume of the box, $Cn^{d\nu}$. Overall we get, under $A$,

$$\# \{\text{such boxes}\} \geq cn^{-d\nu} \sum_{C \text{ small}} N(C)^{(d-1)/d}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^{\nu}$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^{\nu}} \leftrightarrow \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{\mathcal{C}} \sqrt{N(\mathcal{C})}$.

Under $A$,

$$\# \{\text{such boxes}\} \geq cn^{-d\nu} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^{\nu}} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^{\nu}$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$.

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Under $A$,

$$\#\{\text{such boxes}\} \geq cn^{-d\nu} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}.$$

Taking expectations gives

$$Cn^{(1-\nu)d} \mathbb{P}(\Lambda_{2n^{\nu}} \leftrightarrow \partial \Lambda_{n/4}) \geq cn^{-d\nu} \mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. $\mathbb{E} \sum_{C \text{ small}} N(C)^{d-1}/d \geq cn^{(1-\nu)(d-1)}$.

$\mathbb{P}(\Lambda_{2n^\nu} \iff \partial \Lambda_{n/4}) \leq Cn^{-d/2+C\nu} + Cn^{-d+C\nu} \mathbb{E} \sum_{C \text{ small}} \sqrt{N(C)}$.

Under $A$,

$$\#\{\text{such boxes}\} \geq cn^{-d\nu} \sum_{C \text{ small}} N(C)^{d-1}/d.$$  

Taking expectations gives

$$Cn^{(1-\nu)d} \mathbb{P}(\Lambda_{2n^\nu} \iff \partial \Lambda_{n/4}) \geq cn^{-d\nu} \mathbb{E} \sum_{C \text{ small}} N(C)^{d-1}/d.$$  

Together these give

$$\mathbb{E} \sum_{C \text{ small}} N(C)^{d-1}/d \leq Cn^{d/2+C\nu} + Cn^{C\nu} \sum_{C} \mathbb{E} \sqrt{N(C)}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n\nu} \leftrightarrow \partial\Lambda_n) > Cn^{-d}$.

Proof.

Let $N(C)$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $C$. $\mathbb{E} \sum_C \text{small} N(C)^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$

$$\mathbb{E} \sum_{C \text{ small}} N(C)^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \mathbb{E} \sum_C \sqrt{N(C)}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. 

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$$

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \mathbb{E} \sum \sqrt{N(\mathcal{C})}.$$ 

We may add the requirement “$\mathcal{C}$ small” on the right hand side, as the possible large clusters can only add a factor of $Cn^{d/2+C\nu}$. 


Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

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We may add the requirement “$\mathcal{C}$ small” on the right hand side, as the possible large clusters can only add a factor of $Cn^{d/2+C\nu}$. 
Theorem (Duminil-Copin-K-Tassion, unpublished)

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Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \mathbb{E} \sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})}.$$  

We may add the requirement “$\mathcal{C}$ small” on the right hand side, as the possible large clusters can only add a factor of $Cn^{d/2+C\nu}$. Since our clusters all touch both $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$ we must have $N(\mathcal{C}) > cn^{1-\nu}$ for all $\mathcal{C}$. 
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^{\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$

$$\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \mathbb{E} \sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})}.$$

We may add the requirement “$\mathcal{C}$ small” on the right hand side, as the possible large clusters can only add a factor of $Cn^{d/2+C\nu}$. Since our clusters all touch both $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$ we must have $N(\mathcal{C}) > cn^{1-\nu}$ for all $\mathcal{C}$. Thus

$$\sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})} \leq Cn^{-(1-\nu)(d-2)/d} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}.$$
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Proof.

Let $N(\mathcal{C})$ be the number of $n^\nu$-subboxes of $\Lambda_{n/2}$ that intersect $\mathcal{C}$. $\mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \geq cn^{(1-\nu)(d-1)}$

\[ \mathbb{E} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d} \leq Cn^{d/2+C\nu} + Cn^{C\nu} \mathbb{E} \sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})}. \]

We may add the requirement “$\mathcal{C}$ small” on the right hand side, as the possible large clusters can only add a factor of $Cn^{d/2+C\nu}$. Since our clusters all touch both $\Lambda_{n/4}$ and $\partial \Lambda_{n/2}$ we must have $N(\mathcal{C}) > cn^{1-\nu}$ for all $\mathcal{C}$. Thus

\[ \sum_{\mathcal{C} \text{ small}} \sqrt{N(\mathcal{C})} \leq Cn^{-(1-\nu)(d-2)/d} \sum_{\mathcal{C} \text{ small}} N(\mathcal{C})^{(d-1)/d}. \]

For $\nu$ sufficiently small, we reach a contradiction.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

The proof in a nutshell

The Aizenman-Kesten-Newman-Cerf argument gives

$$\mathbb{P}(\Lambda_n^\nu \leftrightarrow \Lambda_n) \leq \text{uninteresting terms } n^{-d} \sum \sqrt{|C|}.$$ 

The contradictory assumption, the isoperimetric inequality and the fact that there are no large clusters give

$$\mathbb{P}(\Lambda_n^\nu \leftrightarrow \Lambda_n) \geq \text{uninteresting terms } n^{-d} \sum |C|(d-1)/d.$$ 

And these two contradict.
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c}(\Lambda_{n^\nu} \leftrightarrow \partial \Lambda_n) > Cn^{-d}$.

Going through the calculation gives

$$\nu < \frac{d - 2}{d^3 + 4d^2 + d - 2}$$

so, say, $1/64$ at $d = 3$. 
Theorem (Duminil-Copin-K-Tassion, unpublished)

For $d \geq 3$ and some $\nu = \nu(d) > 0$, $\mathbb{P}_{p_c} (\Lambda_n^\nu \leftrightarrow \partial \Lambda_n) > C n^{-d}$.

- Going through the calculation gives
  \[ \nu < \frac{d - 2}{d^3 + 4d^2 + d - 2} \]
  so, say, $1/64$ at $d = 3$.

- The theorem holds also at $d = 2$ (known since the 80s, with a different proof).
\begin{align*}
\chi(p_c) &= \infty \\
\sum_{x \in \partial \Lambda_n} \mathbb{P}_{p_c}(0 \leftrightarrow \Lambda_n x) &\geq 1 \\
\mathbb{P}_{p_c}(x \leftrightarrow \Lambda_{2n} y) &> cn^{-C} \\
\mathbb{P}_{p_c} (\Lambda_{nc} \leftrightarrow \Lambda_n) &> cn^{-1/4} \\
\mathbb{P}_{p_c} (\exists \text{ large cluster}) &< 1 - c \\
\mathbb{P}_{p_c} (\Lambda_{nc} \leftrightarrow \partial \Lambda_n) &> Cn^{-d} \\
\mathbb{P}_{p_c} (\text{crossing}) &> c
\end{align*}
For $d \geq 3$, \( \mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \setminus \Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/8} \).
For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/8}$. 

Here

Cerf
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n \stackrel{\Lambda_n \setminus \Lambda_n^c}{\longleftrightarrow} \partial \Lambda_n) \leq Cn^{-1/8}$.

Proof.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. 
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/8}$.

Proof.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n \leftrightarrow \partial \Lambda_n) > cn^{-1/8}$. 
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow_{\Lambda_n^c} \partial \Lambda_n) \leq Cn^{-1/8}$.

Proof.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^{\eta} \leftrightarrow \partial \Lambda_n^\eta) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow_{\Lambda_n^{\gamma}} \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \cap \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).

Lemma

For any $A \supseteq \Lambda_n^\gamma$, $\mathbb{P}(A \leftrightarrow \partial \Lambda_n) \geq P$. 
**Theorem**

For $d \geq 3$, $\Pr(\Lambda_n^c \xleftrightarrow{\Lambda_n \setminus \Lambda_n^c} \partial \Lambda_n) \leq C n^{-1/8}$.

Let $\eta$ be sufficiently small so that $\Pr(\Lambda_n^\eta \Leftrightarrow \Lambda_n) \leq C n^{-1/4}$. Let $\gamma$ be sufficiently small so that $\Pr(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > c n^{-1/8}$. Denote $P = \Pr(\Lambda_n^\gamma \xleftrightarrow{\Lambda_n \setminus \Lambda_n^\gamma} \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).

**Lemma**

For any $A \supseteq \Lambda_n^\gamma$, $\Pr(A \xleftrightarrow{\Lambda_n \setminus A} \partial \Lambda_n) \geq P$. 
**Theorem**

For $d \geq 3$, $\mathbb{P}(\Lambda_n^{c} \xleftrightarrow{\Lambda_n/\Lambda_n^{c}} \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^{\eta} \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow \partial \Lambda_n^{\eta}) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^{\gamma} \xleftrightarrow{\Lambda_n/\Lambda_n^{\gamma}} \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).

**Lemma**

For any $A \supseteq \Lambda_n^{\gamma}$, $\mathbb{P}(A \xleftrightarrow{\Lambda_n/A} \partial \Lambda_n) \geq P$.

Let $\Lambda_n^{\gamma} \subseteq B \subseteq \Lambda_n^{\eta-1}$ and condition on $B = \mathcal{C}(\Lambda_n^{\gamma})$. 
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \setminus \Lambda_n^c \searrow \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \Lambda_n \setminus \Lambda_n^\gamma \searrow \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).

Lemma

For any $A \supseteq \Lambda_n^\gamma$, $\mathbb{P}(A \leftrightarrow \Lambda_n \setminus A \searrow \partial \Lambda_n) \geq P$.

Let $\Lambda_n^\gamma \subseteq B \subseteq \Lambda_n^{\eta - 1}$ and condition on $B = \mathcal{C}(\Lambda_n^\gamma)$. Let $A = \overline{B}$. Outside $A$, the conditioning has no effect.
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \setminus \Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\gamma) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n)$ (i.e. we need to show that $P$ is small).

Lemma

For any $A \supseteq \Lambda_n^\gamma$, $\mathbb{P}(A \leftrightarrow \partial \Lambda_n) \geq P$.

Let $\Lambda_n^\gamma \subseteq B \subseteq \Lambda_n^{\eta - 1}$ and condition on $B = \mathcal{C}(\Lambda_n^\gamma)$. Let $A = \overline{B}$. Outside $A$, the conditioning has no effect. Use the lemma and get

$$\mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma), A \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma)).$$
For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n)$. Let $\Lambda_n^\gamma \subseteq B \subseteq \Lambda_n^{\eta - 1}$ and condition on $B = \mathcal{C}(\Lambda_n^\gamma)$. Let $A = \overline{B}$. Then

$\mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma), A \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma))$. 
For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \ \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^{\eta} \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow \partial \Lambda_n^{\eta}) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow \partial \Lambda_n)$. Let $\Lambda_n^{\gamma} \subseteq B \subseteq \Lambda_n^{\eta-1}$ and condition on $B = \mathcal{C}(\Lambda_n^{\gamma})$. Let $A = \overline{B}$. Then

$\mathbb{P}(B = \mathcal{C}(\Lambda_n^{\gamma}), A \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_n^{\gamma}))$. Sum over all such $B$ and get

$\mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow \partial \Lambda_n^{\eta}, \overline{\mathcal{C}(\Lambda_n^{\gamma})} \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_n^{\gamma} \leftrightarrow \partial \Lambda_n^{\eta})$. 

But the left-hand side implies $\Lambda_n^{\eta} \leftrightarrow \partial \Lambda_n$. 
Theorem

For \( d \geq 3 \), \( \mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \setminus \Lambda_{n \gamma} \setminus \partial \Lambda_n) \leq C n^{-1/8} \).

Let \( \eta \) be sufficiently small so that \( \mathbb{P}(\Lambda_{n \eta} \leftrightarrow \Lambda_n) \leq C n^{-1/4} \). Let \( \gamma \) be sufficiently small so that \( \mathbb{P}(\Lambda_{n \gamma} \leftrightarrow \partial \Lambda_{n \eta}) > c n^{-1/8} \). Denote \( P = \mathbb{P}(\Lambda_{n \gamma} \leftrightarrow \partial \Lambda_n) \). Let \( \Lambda_{n \gamma} \subseteq B \subseteq \Lambda_{n \eta} - 1 \) and condition on \( B = \mathcal{C}(\Lambda_{n \gamma}) \). Let \( A = \overline{B} \). Then

\[
\mathbb{P}(B = \mathcal{C}(\Lambda_{n \gamma}), A \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_{n \gamma})).
\]

Sum over all such \( B \) and get

\[
\mathbb{P}(\Lambda_{n \gamma} \leftrightarrow \partial \Lambda_{n \eta}, \mathcal{C}(\Lambda_{n \gamma}) \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_{n \gamma} \leftrightarrow \partial \Lambda_{n \eta})
\]

But the left-hand side implies \( \Lambda_{n \eta} \leftrightarrow \partial \Lambda_n \).
\[ \Lambda_n \gamma \subset \mathcal{C}(\Lambda_n \gamma) \subset \Lambda_n \eta \subset \Lambda_n \]
Theorem

For \( d \geq 3 \), \( \mathbb{P}(\Lambda_n^c \leftrightarrow \Lambda_n \, \partial \Lambda_n) \leq Cn^{-1/8} \).

Let \( \eta \) be sufficiently small so that \( \mathbb{P}(\Lambda_n \eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4} \). Let \( \gamma \) be sufficiently small so that \( \mathbb{P}(\Lambda_n \gamma \leftrightarrow \partial \Lambda_n \eta) > cn^{-1/8} \). Denote \( P = \mathbb{P}(\Lambda_n \gamma \leftrightarrow \partial \Lambda_n) \). Let \( \Lambda_n \gamma \subseteq B \subseteq \Lambda_n \eta - 1 \) and condition on \( B = \mathcal{C}(\Lambda_n \gamma) \). Let \( A = \overline{B} \). Then

\[
\mathbb{P}(B = \mathcal{C}(\Lambda_n \gamma), A \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_n \gamma)).
\]

Sum over all such \( B \) and get

\[
\mathbb{P}(\Lambda_n \gamma \leftrightarrow \partial \Lambda_n \eta, \mathcal{C}(\Lambda_n \gamma) \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_n \gamma \leftrightarrow \partial \Lambda_n \eta)
\]

But the left-hand side implies \( \Lambda_n \eta \leftrightarrow \partial \Lambda_n \). So we get

\[
Cn^{-1/4} \geq \mathbb{P}(\Lambda_n \eta \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_n \gamma \leftrightarrow \partial \Lambda_n \eta) > cP \cdot n^{-1/8}
\]
Theorem

For $d \geq 3$, $\mathbb{P}(\Lambda_n^c \xleftrightarrow{\Lambda_n \setminus \Lambda_n^c} \partial \Lambda_n) \leq Cn^{-1/8}$.

Let $\eta$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\eta \leftrightarrow \Lambda_n) \leq Cn^{-1/4}$. Let $\gamma$ be sufficiently small so that $\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > cn^{-1/8}$. Denote $P = \mathbb{P}(\Lambda_n^\gamma \xleftrightarrow{\Lambda_n \setminus \Lambda_n^\gamma} \partial \Lambda_n)$. Let $\Lambda_n^\gamma \subseteq B \subseteq \Lambda_n^{\eta-1}$ and condition on $B = \mathcal{C}((\Lambda_n^\gamma))$. Let $A = \overline{B}$. Then

$\mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma), A \xleftrightarrow{\Lambda_n \setminus A} \partial \Lambda_n) \geq P \cdot \mathbb{P}(B = \mathcal{C}(\Lambda_n^\gamma))$. Sum over all such $B$ and get

$\mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta, \mathcal{C}(\Lambda_n^\gamma) \xleftrightarrow{\Lambda_n \setminus \mathcal{C}(\Lambda_n^\gamma)} \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta)$

But the left-hand side implies $\Lambda_n^\eta \leftrightarrow \partial \Lambda_n$. So we get

$Cn^{-1/4} \geq \mathbb{P}(\Lambda_n^\eta \leftrightarrow \partial \Lambda_n) \geq P \cdot \mathbb{P}(\Lambda_n^\gamma \leftrightarrow \partial \Lambda_n^\eta) > cP \cdot n^{-1/8}$

or $P < Cn^{-1/8}$. $\square$
Theorem (Chayes, Chayes, Newman, Grimmett, Kesten, Schonmann...)

For $p < p_c$ there is a number, denoted by $\xi(p)$, such that

$$\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n) = e^{-\left(\xi(p) + o(1)\right)n}.$$
Theorem (Chayes, Chayes, Newman, Grimmett, Kesten, Schonmann...)

For $p < p_c$ there is a number, denoted by $\xi(p)$, such that

$$\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n) = e^{-(\xi(p) + o(1))n}.$$ 

For $p > p_c$ there is a number, also denoted by $\xi(p)$, such that

$$\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n, 0 \leftrightarrow \infty) = e^{-(\xi(p) + o(1))n}.$$ 

The notation $A \leftrightarrow \infty$ means $|\mathcal{C}(A)| = \infty$. 
Theorem (Chayes, Chayes, Newman, Grimmett, Kesten, Schonmann...)

For \( p < p_c \) there is a number, denoted by \( \xi(p) \), such that

\[
\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n) = e^{-\left(\xi(p) + o(1)\right)n}.
\]

For \( p > p_c \) there is a number, also denoted by \( \xi(p) \), such that

\[
\mathbb{P}_p(0 \leftrightarrow \partial \Lambda_n, 0 \not\leftrightarrow \infty) = e^{-\left(\xi(p) + o(1)\right)n}.
\]

The notation \( A \leftrightarrow \infty \) means \( |\mathcal{C}(A)| = \infty \).

Theorem (Duminil-Copin-K-Tassion)

\[
\xi(p) \leq e^{\left|p-p_c\right|^2}.
\]
Theorem (Chayes, Chayes, Newman, Grimmett, Kesten, Schonmann...)

For \( p < p_c \) there is a number, denoted by \( \xi(p) \), such that

\[
P_p(0 \leftrightarrow \partial \Lambda_n) = e^{-(\xi(p)+o(1))n}.
\]

For \( p > p_c \) there is a number, also denoted by \( \xi(p) \), such that

\[
P_p(0 \leftrightarrow \partial \Lambda_n, 0 \leftrightarrow \infty) = e^{-(\xi(p)+o(1))n}.
\]

The notation \( A \leftrightarrow \infty \) means \( |\mathcal{C}(A)| = \infty \).

Theorem (Duminil-Copin-K-Tassion)

\[ \xi(p) \leq e^{\left|p-p_c\right|^{-2}}. \]

We will only show a lemma from proof, to demonstrate yet another use of Cerf’s theorem.
Lemma

If \( \theta := \mathbb{P}(0 \leftrightarrow \infty) > 0 \)

The notation \( A \leftrightarrow \infty \) means \( |\mathcal{C}(A)| = \infty \).
Lemma

If \( \theta := \mathbb{P}(0 \leftrightarrow \infty) > 0 \) then for every \( \varepsilon > 0 \) there exists an \( n \) such that for any set \( A \subseteq \Lambda_n \) intersecting both \( \{0\} \) and \( \partial \Lambda_n \) we have \( \mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon \).

The notation \( A \leftrightarrow \infty \) means \( |\mathcal{C}(A)| = \infty \).
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Proof.

Let \( m \) be such that \( (1 - \theta)^m < \frac{1}{3} \varepsilon \).
**Lemma**

If $\theta := \mathbb{P}(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $\mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon$.

**Proof.**

Let $m$ be such that $(1 - \theta)^m < \frac{1}{3} \varepsilon$. Let $k$ be so large such that

$$\mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}.$$
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Let $K$ be so large that

$$\mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}.$$
Lemma

If $\theta := \mathbb{P}(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $\mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon$.

Proof.

Let $m$ be such that $(1 - \theta)^m < \frac{1}{3}\varepsilon$. Let $k$ be so large such that

$$\mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}.$$ 

Let $K$ be so large that

$$\mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}.$$ 

Define $n = 2Km$. 
Lemma

If $\theta := P(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $P(A \leftrightarrow \infty) > 1 - \varepsilon$.

Proof.

Let $m$ be such that $(1 - \theta)^m < \frac{1}{3}\varepsilon$. Let $k$ be so large such that

$$P(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}.$$ 

Let $K$ be so large that

$$P(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}.$$ 

Define $n = 2Km$. We are now given an $A \subseteq \Lambda_n$. 

Lemma

If \( \theta := \mathbb{P}(0 \leftrightarrow \infty) > 0 \) then for every \( \varepsilon > 0 \) there exists an \( n \) such that for any set \( A \subseteq \Lambda_n \) intersecting both \( \{0\} \) and \( \partial \Lambda_n \) we have \( \mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon \).

Proof.

Let \( m \) be such that \( (1 - \theta)^m < \frac{1}{3} \varepsilon \). Let \( k \) be so large such that

\[
\mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}.
\]

Let \( K \) be so large that

\[
\mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}.
\]

Define \( n = 2Km \). We are now given an \( A \subseteq \Lambda_n \). Find \( m \) elements \( a_1, \ldots, a_m \in A \) such that the translates \( a_i + \Lambda_K \) are disjoint.
Lemma

If $\theta := \mathbb{P}(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $\mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon$. 
Lemma

If \( \theta := \mathbb{P}(0 \leftrightarrow \infty) > 0 \) then for every \( \varepsilon > 0 \) there exists an \( n \) such that for any set \( A \subseteq \Lambda_n \) intersecting both \( \{0\} \) and \( \partial \Lambda_n \) we have \( \mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon \).

Proof.

Let \( m \) be such that \( (1 - \theta)^m < \frac{1}{3}\varepsilon \). Let \( k \) be so large such that \( \mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m} \). Let \( K \) be so large that \( \mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m} \). Define \( n = 2Km \). We are now given an \( A \subseteq \Lambda_n \). Find \( m \) elements \( a_1, \ldots, a_m \in A \) such that the translates \( a_i + \Lambda_K \) are disjoint. For each \( a_i \), \( \mathbb{P}(a_i \leftrightarrow a_i + \partial \Lambda_K) \geq \theta \).
Lemma

If $\theta := \mathbb{P}(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $\mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon$.

Proof.

Let $m$ be such that $(1 - \theta)^m < \frac{1}{3}\varepsilon$. Let $k$ be so large such that $\mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}$. Let $K$ be so large that $\mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}$. Define $n = 2Km$. We are now given an $A \subseteq \Lambda_n$. Find $m$ elements $a_1, \ldots, a_m \in A$ such that the translates $a_i + \Lambda_K$ are disjoint. For each $a_i$, $\mathbb{P}(a_i \leftrightarrow a_i + \partial \Lambda_K) \geq \theta$. Since the boxes are disjoint these are independent and we have

$$\mathbb{P}(\exists i : a_i \leftrightarrow a_i + \partial \Lambda_K) \geq 1 - (1 - \theta)^m > 1 - \frac{\varepsilon}{3}.$$
Lemma

If \( \theta := \mathbb{P}(0 \leftrightarrow \infty) > 0 \), then for every \( \varepsilon > 0 \) there exists an \( n \) such that for any set \( A \subseteq \Lambda_n \) intersecting both \( \{0\} \) and \( \partial \Lambda_n \) we have \( \mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon \).

Proof.

Let \( m \) be such that \( (1 - \theta)^m < \frac{1}{3} \varepsilon \). Let \( k \) be so large such that
\[
\mathbb{P}(\Lambda_k \leftrightarrow \infty) \geq 1 - \frac{\varepsilon}{3m}.
\]
Let \( K \) be so large that
\[
\mathbb{P}(\Lambda_k \leftrightarrow \partial \Lambda_K) < \frac{\varepsilon}{3m}.
\]
Define \( n = 2 Km \). We are now given an \( A \subseteq \Lambda_n \). Find \( m \) elements \( a_1, \ldots, a_m \in A \) such that the translates \( a_i + \Lambda_K \) are disjoint. For each \( a_i \),
\[
\mathbb{P}(a_i \leftrightarrow a_i + \partial \Lambda_K) \geq \theta.
\]
Since the boxes are disjoint these are independent and we have
\[
\mathbb{P}(\exists i : a_i \leftrightarrow a_i + \partial \Lambda_K) \geq 1 - (1 - \theta)^m > 1 - \frac{\varepsilon}{3}.
\]
On the other hand
\[
\mathbb{P}(\forall i : a_i + \Lambda_k \leftrightarrow \infty, a_i + \Lambda_k \not\leftrightarrow a_i + \Lambda_K) > 1 - \frac{2\varepsilon}{3}.
\]
Lemma

If $\theta := \mathbb{P}(0 \leftrightarrow \infty) > 0$ then for every $\varepsilon > 0$ there exists an $n$ such that for any set $A \subseteq \Lambda_n$ intersecting both $\{0\}$ and $\partial \Lambda_n$ we have $\mathbb{P}(A \leftrightarrow \infty) > 1 - \varepsilon$. 
Thanks for your attention!