Schramm-Loewner evolutions and imaginary geometry

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Outline

- Lecture 1: Definition and basic properties of SLE, examples
- Lecture 2: Basic properties of SLE
- Lecture 3: Imaginary geometry

References:

*Conformally invariant processes in the plane* by Lawler
*Lectures on Schramm-Loewner evolution* by Berestycki and Norris
*Imaginary geometry I* by Miller and Sheffield
Simple random walk
Simple random walk
Simple random walk
Donsker’s theorem: Simple random walk converges to Brownian motion.
Loop-erased random walk (LERW)
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Loop-erased random walk (LERW)
Lawler-Schramm-Werner’04: Loop-erased random walk $\Rightarrow$ SLE$_2$.

Illustration by P. Nolin
Smirnov’01: Critical percolation on the triangular lattice $\Rightarrow \text{SLE}_6$
Smirnov’01: Critical percolation on the triangular lattice \( \Rightarrow \text{SLE}_6 \)
$\mathbb{Z}^2$ restricted to a box
Uniform spanning tree (UST)
Uniform spanning tree (UST)

UST with wired $ab$ boundary arc
Uniform spanning tree (UST)

Peano curve

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Uniform spanning tree (UST)

Peano curve

Lawler-Schramm-Werner’04: Peano curve of the UST \( \Rightarrow \text{SLE}_8 \)
Conformal maps

Definition (Conformal map)

$f$ is conformal if $f$ is bijective and $f'$ exists.

$$f(z) = f_1(z_1, z_2) + if_2(z_1, z_2), \quad z = z_1 + iz_2$$

Lemma (Cauchy-Riemann equations)

If $f$ is conformal then

$$\partial_1 f_1 = \partial_2 f_2, \quad \partial_2 f_1 = -\partial_1 f_2.$$
Theorem

- Let $W$ be a planar Brownian motion started from 0.
- Define $\tau_D := \inf\{ t \geq 0 : W(t) \notin D \}$ for $D \subset \mathbb{C}$ a domain s.t. $0 \in D$.
- Let $f : D \to \tilde{D}$ be a conformal map fixing the origin.
- Then $\tilde{W} := f \circ W|_{[0, \tau_D]}$ has the law of a planar Brownian motion run until first leaving $\tilde{D}$, modulo time reparametrization.\(^a\)

\(^a\)We identify $w_1 : l_1 \to \mathbb{C}$ and $w_2 : l_2 \to \mathbb{C}$ (with $l_1, l_2 \subset \mathbb{R}$ intervals) if there is an increasing bijection $\phi : l_1 \to l_2$ such that $w_1 = w_2 \circ \phi$. 
Conformal invariance of planar Brownian motion
Conformal invariance of Brownian motion: proof sketch

Theorem

$\tilde{W} := f \circ W|_{[0, \tau_D]}$ has the law of a planar Brownian motion run until first leaving $\tilde{D}$, modulo time reparametrization.

Write $\tilde{W}(t) = \tilde{W}_1(t) + i\tilde{W}_2(t)$.

Exercise: Show that Itô’s formula and the Cauchy-Riemann equations give

- $\tilde{W}_1, \tilde{W}_2$ are local martingales.
- $\langle \tilde{W}_1 \rangle_t = \langle \tilde{W}_2 \rangle_t$ and this function is a.s. strictly increasing in $t$.
- $\langle \tilde{W}_1, \tilde{W}_2 \rangle \equiv 0$.

These properties characterize a planar Brownian motion modulo time change (see e.g. Revuz-Yor).
Theorem (Riemann mapping theorem)

If $D$ is a non-empty simply connected open proper subset of $\mathbb{C}$ then there exists a conformal map $f : D \rightarrow \mathbb{D}$. 

$D \subset \mathbb{C}$
Theorem (Riemann mapping theorem)

If $D$ is a non-empty simply connected open proper subset of $\mathbb{C}$ then there exists a conformal map $f : D \rightarrow \mathbb{D}$.

Three degrees of freedom.
Mapping out function

- \( \eta : [0, \infty) \rightarrow \mathbb{H} \) curve in \( \mathbb{H} \) from 0 to \( \infty \).
Mapping out function

- $\eta : [0, \infty) \to \mathbb{H}$ curve in $\mathbb{H}$ from 0 to $\infty$.
- $K_t = \mathbb{H} \setminus \{\text{unbounded component of } \mathbb{H} \setminus \eta([0, t])\}$. 

![Diagram](image)
• $\eta : [0, \infty) \to \mathbb{H}$ curve in $\mathbb{H}$ from 0 to $\infty$.
• $K_t = \mathbb{H} \setminus \{\text{unbounded component of } \mathbb{H} \setminus \eta([0, t])\}$.
• $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$, $g_t(\infty) = \infty$. 
Mapping out function

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- $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}, \ g_t(\infty) = \infty$.
- $g_t(z) = a_1z + a_0 + a_{-1}z^{-1} + \ldots$ for $a_1, a_0, \ldots \in \mathbb{R}$ near $z = \infty$.
- Show $\tilde{g}_t(z) := -1/g_t(-z^{-1}) = \tilde{a}_1z + \tilde{a}_2z^2 + \ldots$ by Schwarz reflection.

\[\eta|_{[0,t]} \quad g_t \quad \eta(t)\]
Mapping out function

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- Fix $g_t$ by choosing $a_1 = 1, a_0 = 0$. 

![Diagram](image)
Mapping out function

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- $K_t = \mathbb{H} \setminus \{ \text{unbounded component of } \mathbb{H} \setminus \eta([0, t]) \}$.
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- Fix $g_t$ by choosing $a_1 = 1$, $a_0 = 0$.
- $g_t$ is the mapping out function of the hull $K_t$. 

![Diagram of mapping out function](image)
Mapping out function

- $\eta: [0, \infty) \to \mathbb{H}$ curve in $\mathbb{H}$ from 0 to $\infty$.
- $K_t = \mathbb{H} \setminus \{\text{unbounded component of } \mathbb{H} \setminus \eta([0, t])\}$.
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  - Show $\tilde{g}_t(z) := -1/g_t(-z^{-1}) = \tilde{a}_1z + \tilde{a}_2z^2 + \ldots$ by Schwarz reflection.
- Fix $g_t$ by choosing $a_1 = 1, a_0 = 0$.
- $g_t$ is the mapping out function of the hull $K_t$.
- Remark: Any compact $\mathbb{H}$-hull $K$ (i.e., a bounded subset of $\mathbb{H}$ s.t. $\mathbb{H} \setminus K$ is open and simply connected) can be associated with a mapping out function $g: \mathbb{H} \setminus K \to \mathbb{H}$.
Recall: $g_t(z) = z + a_{-1}z^{-1} + a_{-2}z^{-2} + \ldots$

$\text{hcap}(K_t) := a_{-1}$ is the “size” of $K_t$.

**Lemma (additivity)**

$h\text{cap}(K_{t+s}) = h\text{cap}(K_t) + h\text{cap}(g_t(K_{t+s} \setminus K_t))$. 

$g_t = z + a_{-1}z^{-1} + \ldots$

$z \mapsto z + b_{-1}z^{-1} + \ldots$

$g_{t+s} = z + (a_{-1} + b_{-1})z^{-1} + \ldots$
**Half-plane capacity**

Recall: \( g_t(z) = z + a_{-1}z^{-1} + a_{-2}z^{-2} + \ldots \)

\( \text{hcap}(K_t) := a_{-1} \) is the “size” of \( K_t \).

**Lemma (additivity)**

\[
\text{hcap}(K_t + s) = \text{hcap}(K_t) + \text{hcap}(g_t(K_t + s \setminus K_t)).
\]

**Lemma (scaling)**

\[
\text{hcap}(rK_t) = r^2 \text{hcap}(K_t)
\]

Observe that \( \tilde{g}_t(z) := rg_t(z/r) \) is the mapping out function of \( rK_t \) and that

\[
\tilde{g}_t(z) = z + r^2 \text{hcap}(K_t)z^{-1} + \ldots
\]
Half-plane capacity

Recall: $g_t(z) = z + a_{-1}z^{-1} + a_{-2}z^{-2} + \ldots$

$\text{hcap}(K_t) := a_{-1}$ is the “size” of $K_t$.

Lemma (additivity)

$\text{hcap}(K_{t+s}) = \text{hcap}(K_t) + \text{hcap}(g_t(K_{t+s} \setminus K_t))$.

Lemma (scaling)

$\text{hcap}(rK_t) = r^2 \text{hcap}(K_t)$

Convention: Parametrize $\eta$ such that $\text{hcap}(K_t) = 2t$. 
Driving function and Loewner equation

\( \eta \) simple curve in \((\mathbb{H}, 0, \infty)\) parametrized by half-plane capacity.

**Definition (Driving function)**

\[
W(t) := g_t(\eta(t))
\]

**Proposition (Loewner equation)**

If \( \tau_z = \inf \{ t \geq 0 : z \in K_t \} \) then

\[
\dot{g}_t(z) = \frac{2}{g_t(z) - W(t)} \quad \text{for} \quad t \in [0, \tau_z), \quad g_0(z) = z \in \mathbb{H}.
\]
Schramm’s idea

- Key idea: study $W$ instead of $\eta$.
- If $\eta$ describes the conjectural scaling limit of certain discrete models, then $W$ must be a multiple of a Brownian motion!
Definition of $\text{SLE}_\kappa$ in $(\mathbb{H}, 0, \infty)$

- $\kappa \geq 0$ and $(B(t))_{t \geq 0}$ is a standard Brownian motion.
- Solve Loewner equation with driving function $W = \sqrt{\kappa}B$

$$\dot{g}_t(z) = \frac{2}{g_t(z) - W(t)}, \quad \tau_z = \sup\{t \geq 0 : g_t(z) \text{ well-defined}\}.$$ 

- Define $K_t := \{z \in \mathbb{H} : \tau_z \leq t\}$.
- Let $\eta$ be the curve generating $(K_t)_{t \geq 0}$.
  - $K_t = \mathbb{H} \setminus \{\text{unbounded component of } \mathbb{H} \setminus \eta([0, t])\}$,
  - $\eta$ is well-defined: Rohde-Schramm’05, Lawler-Schramm-Werner’04.

Definition (The Schramm-Loewner evolution in $(\mathbb{H}, 0, \infty)$)

$\eta$ is an $\text{SLE}_\kappa$ in $(\mathbb{H}, 0, \infty)$.

$$(B(t))_{t \geq 0} \rightarrow (g_t)_{t \geq 0} \rightarrow (K_t)_{t \geq 0} \rightarrow (\eta(t))_{t \geq 0}$$
Definition of SLE$_\kappa$ in $(D, a, b)$

**Definition (The Schramm-Loewner evolution)**

- Let $\tilde{\eta}$ be an SLE$_\kappa$ in $(\mathbb{H}, 0, \infty)$.

- Then $\eta := f(\tilde{\eta})$ is an SLE$_\kappa$ in $(D, a, b)$.

- Note that $f$ is not unique since $f \circ \phi_r$ also sends $(\mathbb{H}, 0, \infty)$ to $(D, a, b)$ if $\phi_r(z) := rz$ for $r > 0$.

- SLE$_\kappa$ in $(D, a, b)$ is still well-defined by scale invariance in law of SLE$_\kappa$ in $(\mathbb{H}, 0, \infty)$ (next slide).
Exercise (Scale invariance of $\text{SLE}_\kappa$)

- Let $\eta$ be an $\text{SLE}_\kappa$ in $(\mathbb{H}, 0, \infty)$ and let $r > 0$.
- Prove that $t \mapsto r\eta(t/r^2)$ has the law of an $\text{SLE}_\kappa$ in $(\mathbb{H}, 0, \infty)$. 

Hint: Let $\tilde{\eta}(t) = r\eta(t/r^2)$ and argue that mapping function $\tilde{g}$ of $\tilde{\eta}$ satisfies $\tilde{g}(z) = rg(t/r^2)(z/r^2)$, $\dot{\tilde{g}}(z) = \partial_t r\eta(t/r^2)(z/r^2) = 2\tilde{g}(z) - r\mathcal{W}(t/r^2)$. 

$\eta|_{[0,t/r^2]}$ $\tilde{\eta}|_{[0,t]}$ $z \mapsto rz$ $\eta|_{[0,t]}$ $\tilde{\eta}|_{[0,t]}$ $z \mapsto rz$
Exercise (Scale invariance of \( \text{SLE}_\kappa \))

- Let \( \eta \) be an \( \text{SLE}_\kappa \) in \((\mathbb{H}, 0, \infty)\) and let \( r > 0 \).
- Prove that \( t \mapsto r\eta(t/r^2) \) has the law of an \( \text{SLE}_\kappa \) in \((\mathbb{H}, 0, \infty)\).

Hint: Let \( \tilde{\eta}(t) = r\eta(t/r^2) \) and argue that mapping out fcn \( \tilde{g}_t \) of \( \tilde{\eta} \) satisfy

\[
\tilde{g}_t(z) = rg_t/r^2(z/r), \quad \dot{\tilde{g}}_t(z) = \partial_t(rg_t/r^2(z/r)) = \frac{2}{\tilde{g}_t(z) - rW(t/r^2)}.
\]
Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves $\eta$ modulo time reparametrization in $(D, a, b)$ for each simply connected domain $D \subset \mathbb{C}$, $a, b \in \partial D$.\(^1\)

\(^1\)Identify $\eta$ and $\eta \circ \phi$ if $\phi : l_1 \rightarrow l_2$ cts and strictly increasing, $\partial D$ Martin bdy of $D$. 

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Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves $\eta$ modulo time reparametrization in $(D, a, b)$ for each simply connected domain $D \subset \mathbb{C}$, $a, b \in \partial D$.\(^1\)
- Suppose $\eta \sim \mu_{\mathbb{H},0,\infty}$ a.s. generated by Loewner chain.

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Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves $\eta$ modulo time reparametrization in $(D, a, b)$ for each simply connected domain $D \subset \mathbb{C}$, $a, b \in \partial D$.\(^1\)
- Suppose $\eta \sim \mu_{\mathbb{H},0,\infty}$ a.s. generated by Loewner chain.
- **Conformal invariance (CI):** If $\eta \sim \mu_{D,a,b}$ then $\phi \circ \eta$ has law $\mu_{\tilde{D},\tilde{a},\tilde{b}}$.

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\(^1\)Identify $\eta$ and $\eta \circ \phi$ if $\phi : l_1 \to l_2$ cts and strictly increasing, $\partial D$ Martin bdy of $D$. N. Holden (ETH-ITS Zürich)
Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves $\eta$ modulo time reparametrization in $(D, a, b)$ for each simply connected domain $D \subset \mathbb{C}$, $a, b \in \partial D$.\(^1\)
- Suppose $\eta \sim \mu_{\mathbb{H},0,\infty}$ a.s. generated by Loewner chain.
- **Conformal invariance (CI):** If $\eta \sim \mu_{D,a,b}$ then $\phi \circ \eta$ has law $\mu_{\tilde{D},\tilde{a},\tilde{b}}$.
- **Domain Markov property (DMP):** Conditioned on $\eta|_{[0,\tau]}$ for stopping time $\tau$, the rest of the curve $\eta|_{[\tau,\infty)}$ has law $\mu_{D \setminus K_\tau, \eta(t), b}$.

---

\(^1\)Identify $\eta$ and $\eta \circ \phi$ if $\phi : I_1 \to I_2$ cts and strictly increasing. $\partial D$ Martin bdy of $D$. 

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SLE and imaginary geometry

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Conformal invariance and domain Markov property

- Probability measure $\mu_{D,a,b}$ on curves $\eta$ modulo time reparametrization in $(D,a,b)$ for each simply connected domain $D \subset \mathbb{C}$, $a,b \in \partial D$.\(^1\)
- Suppose $\eta \sim \mu_{\mathbb{H},0,\infty}$ a.s. generated by Loewner chain.
- **Conformal invariance (CI):** If $\eta \sim \mu_{D,a,b}$ then $\phi \circ \eta$ has law $\mu_{\tilde{D},\tilde{a},\tilde{b}}$.
- **Domain Markov property (DMP):** Conditioned on $\eta\mid_{[0,\tau]}$ for stopping time $\tau$, the rest of the curve $\eta\mid_{[\tau,\infty)}$ has law $\mu_{D\setminus K_{\tau},\eta(t),b}$.

**Theorem (Schramm’00)**

The following statements are equivalent:

- $\mu_{D,a,b}$ satisfies (CI) and (DMP).
- There is a $\kappa \geq 0$ such that $\mu_{D,a,b}$ is the law of $\text{SLE}_{\kappa}$.

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\(^1\)Identify $\eta$ and $\eta \circ \phi$ if $\phi : l_1 \rightarrow l_2$ cts and strictly increasing. $\partial D$ Martin bdry of $D$. 

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Conformal invariance of percolation
Conformal invariance of percolation

Conformal invariance: If \( \eta \sim \mu_{D,a,b} \) then \( \phi \circ \eta \) has law \( \mu_{\tilde{D},\tilde{a},\tilde{b}} \).
Lecture 1: Definition and basic properties of SLE, examples
Lecture 2: Basic properties of SLE (today)
Lecture 3: Imaginary geometry

References:
Conformally invariant processes in the plane by Lawler
Lectures on Schramm-Loewner evolution by Berestycki and Norris
Imaginary geometry I by Miller and Sheffield

Key message today: The Loewner equation allows us to analyze SLE using stochastic calculus.
Domain Markov property of percolation
Conditioned on $\eta|_{[0,25]}$, the rest of the percolation interface has the law of a percolation interface in $(D \setminus K_{25}, \eta(25), b)$. 
Number of length $n$ self-avoiding paths on $\mathbb{Z}^2$ from $(0,0)$: $\mu^{n(1+o(1))}$. 
Domain Markov property of the self-avoiding walk

- Number of length $n$ self-avoiding paths on $\mathbb{Z}^2$ from $(0, 0)$: $\mu^{n(1+o(1))}$.
- $\mu \in [2.62, 2.68]$ is the connective constant of $\mathbb{Z}^2$. 

![Diagram of a self-avoiding walk from (0, 0)]
Domain Markov property of the self-avoiding walk

- Number of length $n$ self-avoiding paths on $\mathbb{Z}^2$ from $(0, 0)$: $\mu^{n(1+o(1))}$.
- $\mu \in [2.62, 2.68]$ is the **connective constant** of $\mathbb{Z}^2$.
- **The self-avoiding walk (SAW):** $\mathcal{W}$ random path s.t. for $w$ a self-avoiding path on discrete approximation $(D_m, a_m, b_m)$ to $(D, a, b)$,
  \[
  \mathbb{P}[\mathcal{W} = w] = c\mu^{-|w|},
  \]

  where $|w|$ is the length of $w$ and $c$ is a renormalizing constant.
Domain Markov property of the self-avoiding walk

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\mathbb{P}[\mathcal{W} = w] = c \mu^{-|w|},
$$

where $|w|$ is the length of $w$ and $c$ is a renormalizing constant.
- Conjecture: $\mathcal{W} \Rightarrow SLE_{8/3}$.
Domain Markov property of the self-avoiding walk

- Number of length \( n \) self-avoiding paths on \( \mathbb{Z}^2 \) from \((0, 0)\): \( \mu^{n(1+o(1))} \).
- \( \mu \in [2.62, 2.68] \) is the **connective constant** of \( \mathbb{Z}^2 \).
- **The self-avoiding walk (SAW):** \( \mathcal{W} \) random path s.t. for \( w \) a self-avoiding path on discrete approximation \((D_m, a_m, b_m)\) to \((D, a, b)\),

\[
\mathbb{P}[\mathcal{W} = w] = c\mu^{-|w|},
\]

where \(|w|\) is the length of \( w \) and \( c \) is a renormalizing constant.
- Conjecture: \( \mathcal{W} \Rightarrow \text{SLE}_{8/3} \).
- Exercise: Given \( \mathcal{W}|_{[0,k]} \) the remaining path has the law of a SAW in \((D_m \setminus \mathcal{W}([0, k]), \mathcal{W}(k), b_m)\).
SLE satisfies (CI) and (DMP)

- (CI): follows from the definition of $\text{SLE}_\kappa$ on general domains $(D, a, b)$. 

![Diagram showing SLE on general domains](image)
SLE satisfies (CI) and (DMP)

- (CI): follows from the definition of SLE$_\kappa$ on general domains $(D, a, b)$.
- (DMP): sufficient to verify for $(\mathbb{H}, 0, \infty)$ and parametrization by half-plane capacity.

Want to prove: $\eta |_{[\tau, \infty)}$ has the law of an SLE$_\kappa$ in $(\mathbb{H} \setminus K_\tau, \eta(\tau), \infty)$. 
SLE satisfies (CI) and (DMP)

- (CI): follows from the definition of SLE\(_\kappa\) on general domains \((D, a, b)\).
- (DMP): sufficient to verify for \((\mathbb{H}, 0, \infty)\) and parametrization by half-plane capacity.

Centered mapping out functions \(\tilde{g}_t(z) := g_t(z) - W(t)\) satisfy

\[
d\tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z)} - dW(t), \quad \tilde{g}_0(z) = z. \tag{CL}
\]

Exercise: Centered mapping out functions \((\tilde{g}_\tau, t)_{t \geq 0}\) of \(\hat{\eta}^\tau\) satisfy

\[
\tilde{g}_{\tau+t} = \tilde{g}_{\tau, t} \circ \tilde{g}_\tau.
\]

Exercise: Use previous exercise to argue that \((\tilde{g}_\tau, t)_{t \geq 0}\) satisfies (CL)

w/driving function \((W(\tau + t) - W(\tau))_{t \geq 0} \overset{d}{=} (W(t))_{t \geq 0}\).

The last exercise implies that \(\hat{\eta}^\tau\) has the law of an SLE\(_\kappa\) in \((\mathbb{H}, 0, \infty)\).
(CI) and (DMP) imply that $\eta$ is an SLE

- Suppose $(\mu_{D,a,b})_{D,a,b}$ satisfies (CI) and (DMP). Let $\eta \sim \mu_{\mathbb{H},0,\infty}$ be param. by half-plane capacity; let $W$ denote the driving fcn of $\eta$. 
(CI) and (DMP) imply that $\eta$ is an SLE

- Suppose $(\mu_{D,a,b})_{D,a,b}$ satisfies (CI) and (DMP). Let $\eta \sim \mu_{\mathbb{H},0,\infty}$ be param. by half-plane capacity; let $W$ denote the driving fcn of $\eta$.
- $(\text{CI}) \Rightarrow \text{scale invariance} \Rightarrow (W(t))_{t \geq 0} \overset{d}{=} (rW(t/r^2))_{t \geq 0}$.

![Diagram of SLE and imaginary geometry](image)

Let $\tilde{\eta}(t) := r\eta(t/r^2)$. Then $\eta \overset{d}{=} \tilde{\eta}$. Mapping out fcn $(\tilde{g}_t)_{t \geq 0}$ of $\tilde{\eta}$ satisfy:

$$\tilde{g}_t(z) = rg_{t/r^2}(z/r), \quad \dot{\tilde{g}}_t(z) = \partial_t (rg_{t/r^2}(z/r)) = \frac{2}{\tilde{g}_t(z) - rW(t/r^2)}.$$
(CI) and (DMP) imply that $\eta$ is an SLE

- Suppose $(\mu_{D,a,b})_{D,a,b}$ satisfies (CI) and (DMP). Let $\eta \sim \mu_{\mathbb{H},0,\infty}$ be param. by half-plane capacity; let $W$ denote the driving fcn of $\eta$.
- (CI) $\Rightarrow$ scale invariance $\Rightarrow (W(t))_{t \geq 0} \overset{d}{=} (rW(t/r^2))_{t \geq 0}$.
- (DMP)

$$(DMP): \eta|_{[s,\infty)} \text{ has law } \mu_{\mathbb{H}\setminus K_s,\eta(s),\infty}.$$
(CI) and (DMP) imply that \( \eta \) is an SLE

- Suppose \((\mu_{D,a,b})_{D,a,b}\) satisfies (CI) and (DMP). Let \( \eta \sim \mu_{\mathbb{H},0,\infty} \) be param. by half-plane capacity; let \( W \) denote the driving fcn of \( \eta \).

- (CI) \( \Rightarrow \) scale invariance \( \Rightarrow \) \((W(t))_{t \geq 0} \overset{d}{=} (rW(t/r^2))_{t \geq 0}\).

- (DMP) \( \Rightarrow \) \((W(t))_{t \geq 0}\) has i.i.d. increments.

- By (DMP), \( \hat{\eta}^s \overset{d}{=} \eta \) and \( \hat{\eta}^s \) is independent of \( \eta|_{[0,s]} \).

- The centered mapping out fcn \((\hat{g}_s,t)_{t \geq 0}\) of \( \hat{\eta}^s \) satisfy the centered Loewner equation w/driving function \((W(s + t) - W(s))_{t \geq 0}\).

- Combining the above, \((W(s + t) - W(s))_{t \geq 0} \overset{d}{=} (W(t))_{t \geq 0}\) and is independent of \( W|_{[0,s]} \).
(CI) and (DMP) imply that $\eta$ is an SLE

- Suppose $(\mu_{D,a,b})_{D,a,b}$ satisfies (CI) and (DMP). Let $\eta \sim \mu_{\mathbb{H},0,\infty}$ be param. by half-plane capacity; let $W$ denote the driving fcn of $\eta$.
- (CI) $\Rightarrow$ scale invariance $\Rightarrow (W(t))_{t \geq 0} \overset{d}{=} (rW(t/r^2))_{t \geq 0}$.
- (DMP) $\Rightarrow (W(t))_{t \geq 0}$ has i.i.d. increments.
- (CI) + (DMP) $\Rightarrow W = \sqrt{\kappa}B$ for some $\kappa \geq 0$. 
Phases of SLE

Rohde-Schramm’05: SLE$_\kappa$ has the following phases:

- $\kappa \in [0, 4]$: The curve is simple.
- $\kappa \in (4, 8)$: The curve is self-intersecting and has zero Lebesgue measure.
- $\kappa \geq 8$: The curve fills space.

Figures by P. Nolin, W. Werner, and J. Miller
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Phase transition at $\kappa = 4$

Lemma

- If $\kappa \in [0, 4]$ then $\eta$ is a.s. simple (i.e., $\eta(t_1) \neq \eta(t_2)$ for $t_1 \neq t_2$).
- If $\kappa > 4$ then $\eta$ is a.s. not simple.

We will deduce the lemma from the following result, where

$$\tau_x = \inf\{t \geq 0 : x \in \overline{K_t}\} \text{ for } x > 0.$$

Lemma

- If $\kappa \in [0, 4]$ then $\tau_x = \infty$ a.s.
- If $\kappa > 4$ then $\tau_x < \infty$ a.s.
Phase transition at $\kappa = 4$

Recall $\tau_x = \inf\{t \geq 0 : x \in \overline{K}_t\}$ for $x > 0$.

**Lemma**

- If $\kappa \in [0, 4]$ then $\tau_x = \infty$ a.s.
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w.l.o.g. $x = 1$; \[ \dot{g}_t(1) = \frac{2}{g_t(1) - \sqrt{\kappa}B(t)}, \quad g_t(1) = 1, \]
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$$Y(t) = \kappa^{-1/2}(g_t(1) - \sqrt{\kappa}B(t)),$$
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\[ \tau_1 = \inf\{t \geq 0 : Y(t) = 0\}, \]
\[ dY(t) = \frac{2}{\kappa Y(t)} dt - dB(t), \] so $Y(t)$ is a $\left(\frac{4}{\kappa} + 1\right)$-dim. Bessel process.
Phase transition at $\kappa = 4$

**Lemma**
- If $\kappa \in [0, 4]$ then $\eta$ is a.s. simple (i.e., $\eta(t_1) \neq \eta(t_2)$ for $t_1 \neq t_2$).
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- If $\kappa \in [0, 4]$ then $\tau_x = \infty$ a.s.
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**Proposition**

- \( \eta \) SLE\(_6\) in \((D, x, y)\). Set \( \tau := \inf\{t \geq 0 : \eta(t) \in \text{arc}(\tilde{y}, y)\} \).
- Define \( \tilde{\eta} \) and \( \tilde{\tau} \) in the same way for \((D, x, \tilde{y})\).
- Then \( \eta|_{[0, \tau]} \overset{d}{=} \tilde{\eta}|_{[0, \tilde{\tau}]} \).
Locality of SLE$_6$

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- Then $\eta|_{[0, \tau]} \overset{d}{=} \tilde{\eta}|_{[0, \tilde{\tau}]}$.

Want to prove: If $\eta$ is an SLE$_6$ in $(\mathbb{H}, 0, \infty)$ then $\eta$ has the law of an SLE$_6$ in $(\mathbb{H}, 0, y)$ until hitting $L$. 

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Locality of SLE\textsubscript{6}: Proof sketch

- \( \eta \) SLE\textsubscript{6} in \((\mathbb{H}, 0, \infty)\); \( g_t \) mapping out function; \( W \) driving function.

\[
\Phi_t := g_t^* \circ \Phi \circ g_t^{-1}
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Localy of SLE$_6$: Proof sketch

- $\eta$ SLE$_6$ in $(\mathbb{H}, 0, \infty)$; $g_t$ mapping out function; $W$ driving function.
- $\Phi$ conformal map sending $(\mathbb{H}, 0, y)$ to $(\mathbb{H}, 0, \infty)$.

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\Phi_t := g^*_t \circ \Phi \circ g^{-1}_t
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$\eta^*$ in $(\mathbb{H}, 0, \infty)$; \(g^*_t\) mapping out function; \(W^*(t)\) driving function.

\[
\dot{g}^*_t(z) = b'(t) g^*_t(z) - W^*(t)
\]

$\dot{g}^*(t) = hcap(\eta^*(\mathbb{H}, 0, t))$. 

Want to show:

- $\eta^*$ law of SLE$_6$ in $(\mathbb{H}, 0, \infty)$ until hitting $L^*$.
- Equivalently, $W^*(t) = \sqrt{6} B^*(b(t)/2)$ for $B^*$ std Brownian motion.

Find $dW^*$ by Itô's formula; prove and use $\dot{\Phi}_t(W(t)) = -3\Phi''(W(t))$. 

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- \( \eta^*(t) := \Phi(\eta(t)) \); \( g_t^* \) map. out fcn; \( W^*(t) = \Phi_t(W(t)) \) driving fcn.

\[ \dot{g}_t^*(z) = \frac{b'(t)}{g_t^*(z) - W^*(t)} , \quad b(t) = \text{hcap}(\eta^*([0, t])). \]
Locality of SLE$_6$: Proof sketch

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- Find $dW^*$ by Itô's formula; prove and use $\dot{\Phi}_t(W(t)) = -3\Phi''(W(t))$. 
Restriction property

**Definition**

- Let $\mu_{D,x,y}$ for $D \subset \mathbb{C}$ simply connected and $x, y \in \partial D$ be a family of probability measures on curves $\eta$ in $D$ from $x$ to $y$.
- Let $\eta \sim \mu_{D,x,y}$ for some $(D, x, y)$ and let $U \subset D$ be simply connected s.t. $x, y \in \partial U$.
- The measures $\mu_{D,x,y}$ satisfy the **restriction property** if $\eta$ conditioned to stay in $U$ has the law of a curve sampled from $\mu_{U,x,y}$.

For which $\kappa \geq 0$ does $\text{SLE}_\kappa$ satisfy the restriction property?
Does the loop-erased random walk satisfy the restriction property?
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Let $\hat{W}$ be a simple random walk on discrete approximation $(D_m, a_m, b_m)$ to $(D, a, b)$.
Does the loop-erased random walk satisfy the restriction property?

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The **loop-erased random walk (LERW)** $W$ is loop-erasure of $\hat{W}$. 
Does the loop-erased random walk satisfy the restriction prop.? NO

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Let $U_m \subset D_m$ be connected s.t. $a_m, b_m \in U_m$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Restriction property of discrete models

- Does the loop-erased random walk satisfy the restriction prop.? NO

- Let \( \hat{W} \) be a simple random walk on discrete approximation \((D_m, a_m, b_m)\) to \((D, a, b)\).
- The loop-erased random walk (LERW) \( W \) is loop-erasure of \( \hat{W} \).
- Let \( U_m \subset D_m \) be connected s.t. \( a_m, b_m \in U_m \).
- “LERW in \((D_m, a_m, b_m)\) conditioned to stay in \( U_m \)” \( \neq \) “LERW in \((U_m, a_m, b_m)\)”, since the latter requires \( \hat{W} \subset U_m \) (not just \( W \subset U_m \)).
Restriction property of discrete models

- Does the loop-erased random walk satisfy the restriction prop.? **NO**
- Does the self-avoiding walk satisfy the restriction property?

The **self-avoiding walk (SAW)** $W$ is s.t. for any fixed self-avoiding path $w$ on discrete approximation $(D_m, a_m, b_m)$ to $(D, a, b)$,

$$\mathbb{P}[W = w] = c\mu^{-|w|},$$

where $\mu$ is the connective constant, $|w|$ is the length of $w$, and $c$ is a renormalizing constant.
Restriction property of discrete models

- Does the loop-erased random walk satisfy the restriction prop.? NO
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The **self-avoiding walk (SAW)** $\mathcal{W}$ is s.t. for any fixed self-avoiding path $w$ on discrete approximation $(D_m, a_m, b_m)$ to $(D, a, b)$,

$$\mathbb{P}[\mathcal{W} = w] = c\mu^{-|w|},$$

where $\mu$ is the connective constant, $|w|$ is the length of $w$, and $c$ is a renormalizing constant.

“SAW in $(D_m, a_m, b_m)$ cond. to stay in $U_m$” $\overset{d}{=} “SAW$ in $(U_m, a_m, b_m)$”
Proposition

- $\eta$ SLE$_{8/3}$ in $(\mathbb{H}, 0, \infty)$; $K \subset \mathbb{H}$ s.t. $\mathbb{H} \setminus K$ simply conn., $0, \infty \notin \overline{K}$.
- Then $\eta$ cond. on $\eta \cap K = \emptyset$ has the law of SLE$_{8/3}$ in $(\mathbb{H} \setminus K, 0, \infty)$. 

![Diagram](image-url)
Proposition

- \( \eta \) \( \text{SLE}_{8/3} \) in \((\mathbb{H}, 0, \infty)\); \( K \subset \mathbb{H} \) s.t. \( \mathbb{H} \setminus K \) simply conn., \( 0, \infty \not\in \overline{K} \).
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Proposition equivalent to the following for \( K' \supset K \)

\[
P[\eta \cap K' = \emptyset \mid \eta \cap K = \emptyset] = P[\eta \cap \tilde{g}_K(K') = \emptyset], \tag{A}
\]

since RHS = \( P[\hat{\eta} \cap K' = \emptyset] \) for \( \hat{\eta} \) an \( \text{SLE}_{8/3} \) in \((\mathbb{H} \setminus K, 0, \infty)\).
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\mathbb{P}[\eta \cap K' = \emptyset | \eta \cap K = \emptyset] = \mathbb{P}[\eta \cap \tilde{g}_K(K') = \emptyset],
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since RHS = \( \mathbb{P}[\hat{\eta} \cap K' = \emptyset] \) for \( \hat{\eta} \) an SLE\(_{8/3} \) in \((\mathbb{H} \setminus K, 0, \infty)\).

- Key identity (proof omitted here): \( \mathbb{P}[\eta \cap K = \emptyset] = \tilde{g}'_K(0)^{5/8} \).

\[\tilde{g}_K = z + a + O(z^{-1})\]
Proposition

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- This identity, Bayes’ rule, and \( \tilde{g}_K(K') = \tilde{g}_{\tilde{g}_K(K')}(K') \circ \tilde{g}_K \) imply (A).

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Proposition

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- Then $\eta$ cond. on $\eta \cap K = \emptyset$ has the law of SLE$_{8/3}$ in $(\mathbb{H} \setminus K, 0, \infty)$.

Proposition equivalent to the following for $K' \supset K$

$$\mathbb{P}[\eta \cap K' = \emptyset \mid \eta \cap K = \emptyset] = \mathbb{P}[\eta \cap \tilde{g}_K(K') = \emptyset], \quad (A)$$

since RHS = $\mathbb{P}[\hat{\eta} \cap K' = \emptyset]$ for $\hat{\eta}$ an SLE$_{8/3}$ in $(\mathbb{H} \setminus K, 0, \infty)$.

- Key identity (proof omitted here): $\mathbb{P}[\eta \cap K = \emptyset] = \tilde{g}'_K(0)^{5/8}$.
- This identity, Bayes’ rule, and $\tilde{g}_{K'} = \tilde{g}_{\tilde{g}_K(K')} \circ \tilde{g}_K$ imply (A).
- Remark: Key identity with exponent $\alpha \geq 5/8$ represent other random sets satisfying conformal restriction.

\[\begin{align*}
\tilde{g}_K &= z + a + O(z^{-1}) \\
\tilde{g}_K(K')&
\end{align*}\]
Chordal, radial, and whole-plane SLE

chordal SLE  radial SLE  whole-plane SLE
A few open questions

- Convergence of discrete models, e.g.
  - self-avoiding walk ($\kappa = 8/3$)
  - universality for percolation: $\mathbb{Z}^2$; Voronoi tessellation ($\kappa = 6$)
  - Fortuin-Kastelyn model ($\kappa \in (4, 8)$)
  - 6-vertex model ($\kappa = 12$, general $\kappa$)
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For each edge

we have

or
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(a) 6-vertex configuration

(b) Peano curve
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Random planar map; figure due to Gwynne-Miller-Sheffield
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  - loop-erased random walk (Kozma’07)
  - uniform spanning tree (Angel–Croydon–Hernandez-Torres–Shiraishi’20)
  - percolation

3d UST; figure by Angel–Croydon–Hernandez-Torres–Shiraishi
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Figure by Sheffield-Yadin
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  - uniform spanning tree (Angel–Croydon–Hernandez-Torres–Shiraishi’20)
  - percolation

- Path properties of SLE, e.g.
  - Hausdorff measure of SLE
Thanks for attending!
Radial SLE

- \( g_t : \mathbb{D} \setminus K_t \to \mathbb{D} \) defined such that \( g_t(0) = 0 \) and \( g'_t(0) > 0 \).
- \( \eta \) parametrized such that \( t = \log g'_t(0) \).
- Radial Loewner equation, where \( B \) is a standard Brownian motion

\[
\dot{g}_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa}B(t)} + g_t(z)}{e^{i\sqrt{\kappa}B(t)} - g_t(z)}, \quad g_0(z) = z.
\]
Radial SLE

- $g_t: \mathbb{D} \setminus K_t \to \mathbb{D}$ defined such that $g_t(0) = 0$ and $g_t'(0) > 0$.
- $\eta$ parametrized such that $t = \log g_t'(0)$.
- Radial Loewner equation, where $B$ is a standard Brownian motion

\[
\dot{g}_t(z) = g_t(z) \frac{e^{i\sqrt{\kappa}B(t)} + g_t(z)}{e^{i\sqrt{\kappa}B(t)} - g_t(z)}, \quad g_0(z) = z.
\]
Conditioned on $\eta|_{(-\infty,t]}$, the remainder $\eta|_{(t,\infty)}$ of the curve has the law of radial SLE$_{\kappa}$ in $(\mathbb{C} \setminus K_t, \eta(t), b)$. 

$K_t = \eta((\infty,t])$

$g_t(b) = 0$