Branching random walks, continuation

BRW with killing/selection/interactions

Example for a BRW with killing: \( p(2) = 1 \)

\[ \Pr[X = 0] = 1 - p \]
\[ \Pr[X = 1] = p \]

ray: \( v_0, v_1, v_2, v_3, \ldots \)

\( v_{i+1} \) child of \( v_i \)

\[ \Delta (\varepsilon, p) = \Pr[ \exists i \cup \text{ray s.t.} \]
\[ S_{v_i} \geq (x^* - \varepsilon)_j, \quad \forall j \geq 1 ] \]
\[ = \Pr[ "\text{nearby optimal ray}" ] \]

Q: \( \Delta (\varepsilon, p) \rightarrow ? \) (for fixed \( p \), as \( \varepsilon \rightarrow 0 \))

* For \( p > \frac{1}{2} \), \( x^* = 1 \), with prob. there is an optimal ray \( \rightarrow \delta (\varepsilon, p) \rightarrow 0 \)
\[ p = \frac{1}{2} \quad \mathbb{P}(\varepsilon, p) \sim c \varepsilon \quad (R, Remantle) \]

Take \( \varepsilon = \frac{1}{n} \) \( u = \frac{1}{\varepsilon} \)

\[ \frac{P}{n} \left[ \text{Every only 1's up to level } u \right] \sim \frac{2}{n} \]

(PL crit, GW process survives up to gen. \( u \) \( \sim \frac{2}{n} \))

- For \( p < \frac{1}{2} \),
  \[ \log \mathbb{P}(\varepsilon, p) \sim \frac{-c(p)}{\sqrt{\varepsilon}} \quad \text{for } \varepsilon \to 0 \]
  (NG, Y. Hu, Z. Shi)

Idea:
Replace large deviations for endpoint with large deviations for the whole path.
**Theorem**

\( \{ \ldots, Y_2, Y_1 \} \) iid \( \mathbb{E} \left[ \frac{1}{i} \right] = 0, \mathbb{E} |Y_1|^2 + \delta < \infty \)

\( S_n = \sum_{i=1}^{n} Y_i \), \( g_1(\cdot), g_2(\cdot) \) cont., \( g_1(0) < 0 < g_2(0) \)

\( g_1 < g_2 \)

\[ \frac{a_n^2}{2} \log \mathbb{P} \left[ \frac{g_1(i)}{a_n} \leq \frac{\sum_{i=1}^{n} Y_i}{a_n} \leq g_2(i/a_n) \right] \]

\[ \rightarrow -J(g_1, g_2, \delta^2) \]

Here \( a_n \to \infty, \frac{a_n}{\sqrt{n}} \to 0 \)

**N-BRW**

Example with selection

- keep only the N individuals with the largest positions

Bouvet / Deuvidal / Moeller / Meunier
\[ M_{n,N} = \max_{v \in D_{n,N}} S_v \]

Then \( \frac{M_{n,N}}{n} \xrightarrow{u \to \infty} x_N^* \) a.s.

Thus J. Bévard, J.-B. Gouezé
\[ p(2) = 1, \quad \mathbb{E}[e^{x^*}] < \infty \quad \forall x \in \mathbb{R} \]
\[ x^* - x_N^* \sim \frac{\text{const}}{(\log N)^2} \quad \text{for } N \to \infty \]

\[ \rightarrow \text{J. Bévard P. Mailléard for heavy-tailed } X, \text{ i.e.} \]
\[ \mathbb{P}[X > t] \sim \frac{1}{t^x} \quad 0 < x < 2 \]

Heuristics: The foll. two events are comparable:

(1) BRW with killing at slope \( x^* - \varepsilon \), starting with \( N \) particles survives

(2) \( N - \text{BRW moves at a speed} \geq x^* - \varepsilon \).

(1) Prob. is \( 1 - (1 - p(\varepsilon))^N \)
\[(2) \quad x^* - x^*_N \text{ should be of order } \varepsilon = \varepsilon(N) \text{ with } \varepsilon \text{ s.t.} \]
\[\delta(\varepsilon) \approx \frac{1}{N} \]

But since \( \delta(\varepsilon) \approx e^{-\frac{c}{\sqrt{\varepsilon}}} \approx \frac{1}{N} \)

have \( \varepsilon \approx \frac{c}{(\log N)^2} \).

L- BRW

- keep only the individuals within (spatial) distance \( L \) to \( M_n \), remove all the others.

\[x^*_L = \lim_{n \to \infty} \frac{M_{u,L}}{n} \quad \text{clearly:} \quad x^*_L \leq x^* \]

Conj. BDMM

\[x^*_L \approx x^*_L = \log N\]

Thus for L-BBM by M. Pain.

BRW with interactions
Take a BRW with some $X \sim N(0,1)$. Fix $R \in \mathbb{R}$

- Kill two particles if they come closer than $R$.

Is the survival probability strictly positive? Open!

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**Fragmentation process**

$(I_t)_{t \geq 0}$

Fix $\alpha > 0$. $I_t$ is collection of disjoint intervals $\in (0,1)$

An interval $(a, b)$ with $u = b - a$ splits into $u$ subintervals at rate $\alpha (a, a + \frac{u}{m}, a + \frac{2u}{m}, \ldots, b)$

$N_t(j) = \#$ intervals $(a, b)$ at time $t \in (a, b - a = m \cdot j$)

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**Multitype branching process**

Types in $\mathbb{N}_0$. Particles die at rate $q_j$, replaced by $m$ offspring of type $j+1$. ($q > 0$ fixed)
\[ N_t(j) = \# \text{particles of type } j \text{ present at time } t \]

- Tree-indexed RW

\[
(\text{We}) \text{ indep.} \quad \text{We} = \mathcal{E} x p(\mu^n)
\]

if \( e \) goes from level \( u-1 \) to level \( u \).

Say \( v \) is alive at time \( t \) if \( S_v \leq t, S_{v_i} > t \) \( \forall 1 \leq i \leq u \).

Study \( N_t = \sum_{n \geq 0} N_t(n) \) \( \forall \) child of \( v \).

- Work in progress with

  P. Duszewski, S. Johnston, J. Puchono, D. Schmid

Thus Buennou / Douweelb

\[ q \leq 1: \mathbb{E} \left[ N_t \right] \sim t^\beta \] where

\[ q = m^{-1} = \frac{t^{-1}}{\alpha}, \quad \beta = \frac{\log m}{\log \frac{1}{q}} \]

\[ \mathbb{E} \left[ N_t \right] = \sum_{e=0}^{\infty} m^e e^{-q e t} \]
\* \( q > 1 \) \quad \text{"explosive"}
\* \( q = 1 \) \quad \text{"classical"}
\* \( q < 1 \) \quad \text{"slow"}