Invariant measures for
KdV and Toda-type
discrete integrable systems

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joint with
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1. KDV AND TODA-TYPE DISCRETE INTEGRABLE SYSTEMS
KDV AND TODA LATTICE EQUATIONS

Korteweg-de Vries (KdV) equation:
\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,
\]
where \( u = (u(x,t))_{x,t \in \mathbb{R}} \).

Toda lattice equation:
\[
\begin{align*}
\frac{d}{dt}p_n &= e^{-(q_n-q_{n-1})} - e^{-(q_n+1-q_n)}, \\
\frac{d}{dt}q_n &= p_n,
\end{align*}
\]
where \( p_n = (p_n(t))_{t \in \mathbb{R}}, q_n = (q_n(t))_{t \in \mathbb{R}} \).
KORTEWEG-DE VRIES (KdV) equation:
\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,
\]
where \( u = (u(x, t))_{x, t \in \mathbb{R}}. \)

TODA LATTICE equation:
\[
\begin{cases}
\frac{d}{dt} p_n &= e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \\
\frac{d}{dt} q_n &= p_n,
\end{cases}
\]
where \( p_n = (p_n(t))_{t \in \mathbb{R}}, q_n = (q_n(t))_{t \in \mathbb{R}}. \)
Box-ball System (BBS)

Discrete time deterministic dynamical system (cellular automaton) introduced in 1990 by Takahashi and Satsuma. In original work, configurations \((\eta_x)_{x \in \mathbb{Z}}\) with a finite number of balls were considered. (NB. Empty box: \(\eta_x = 0\); ball \(\eta_x = 1\).)

- Every ball moves exactly once in each evolution time step.
- The leftmost ball moves first and the next leftmost ball moves next and so on...
- Each ball moves to its nearest right vacant box.

Dynamics \(T: \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}\):

\[
(T\eta)_n = \min \left\{1 - \eta_n, \sum_{m=-\infty}^{n-1} (\eta_m - (T\eta)_m) \right\},
\]

where \((T\eta)_n = 0\) to left of particles.
BBS CARRIER

- Carrier moves **left to right**
- Picks up a ball if it finds one
- Puts down a ball if it comes to an empty box when it carries at least one ball

Set $U_n$ to be number of balls carried from $n$ to $n + 1$, then

$$U_n = \begin{cases} 
U_{n-1} + 1, & \text{if } \eta_n = 1, \\
U_{n-1}, & \text{if } \eta_n = 0, U_{n-1} = 0, \\
U_{n-1} - 1, & \text{if } \eta_n = 0, U_{n-1} > 0, 
\end{cases}$$

and

$$(T\eta)_n = \min \{1 - \eta_n, U_{n-1}\}.$$
The local dynamics of the BBS are described via a system of lattice equations:

\[
\cdots U^t_{n-1} \xrightarrow{F^{(1,\infty)}_{\text{udK}}^t} U^t_n \xrightarrow{F^{(1,\infty)}_{\text{udK}}^t} U^t_{n+1} \cdots,
\]

where \( F^{(1,\infty)}_{\text{udK}} \) is an involution, as given by:

\[
F^{(1,\infty)}_{\text{udK}}(\eta, u) := (\min\{1 - \eta, u\}, \eta + u - \min\{1 - \eta, u\}).
\]

This is (a version of) the ultra-discrete KdV equation (udKdV). Can generalise to box capacity \( J \in \mathbb{N} \cup \{\infty\} \) and carrier capacity \( K \in \mathbb{N} \cup \{\infty\} \).
BASIC QUESTIONS

In today’s talk, I will address two main topics for the BBS (and related systems):

• Existence and uniqueness of solutions to initial value problem for (udKdV) with infinite configurations?

• I.i.d. invariant measures on initial configurations?

Other recent developments in the study of the BBS that I will not talk about:

• Invariant measures based on solitons, e.g. [Ferrari, Nguyen, Rolla, Wang]. See also [Levine, Lyu, Pike], etc.

• Generalized hydrodynamic limits, e.g. [C., Sasada], [Kuniba, Misguich, Pasquier].
INTEGRABLE SYSTEMS DERIVED FROM THE KDV AND TODA EQUATIONS
ULTRA-DISCRETE KDV EQUATION (UDKDV)

<table>
<thead>
<tr>
<th>Model</th>
<th>Lattice structure</th>
<th>Local dynamics: $F_{udK}^{(J,K)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>udKdV</td>
<td>$U_{n-1}^t$ $\eta_{n}^t$ $U_n^t$ $\eta_{n}^{t+1}$</td>
<td>$a + \min{J-a,b}$ $- \min{a,K-b}$ $b + \min{a,K-b}$ $- \min{J-a,b}$</td>
</tr>
</tbody>
</table>

Variables are $\mathbb{R}$-valued. Parameter $J$ represents box capacity, $K$ represents carrier capacity. Multi-coloured version of BBS/UDKDV also studied [Kondo].
## DISCRETE KDV EQUATION (DKDV)

<table>
<thead>
<tr>
<th>Model</th>
<th>Lattice structure</th>
<th>Local dynamics: $F_{dK}^{(\alpha, \beta)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dKdV</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$U_{n-1}^t \rightarrow U_n^t$</td>
<td>$\frac{b(1+\beta ab)}{(1+\alpha ab)}$ \rightarrow $a(1+\alpha ab)$ \rightarrow $b(1+\beta ab)$</td>
</tr>
</tbody>
</table>

Variables are $(0, \infty)$-valued. UDKdV is obtained as ultra-discrete/zero-temperature limit by making change of variables:

$$\alpha = e^{-J/\varepsilon}, \quad \beta = e^{-K/\varepsilon}, \quad a = e^{a/\varepsilon}, \quad b = e^{b/\varepsilon}.$$
# ULTRA-DISCRETE TODA EQUATION (UDTODA)

<table>
<thead>
<tr>
<th>Model</th>
<th>Lattice structure</th>
<th>Local dynamics: $F_{udT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>udToda</td>
<td>$Q_{n+1}^t$ $E_{n+1}^t$</td>
<td>$\min{b, c}$ $a+b - \min{b, c}$</td>
</tr>
<tr>
<td></td>
<td>$U_n^t$ $E_n^t$ $Q_{n+1}^t$</td>
<td>$c$ $b$ $a$ $a+c - \min{b, c}$</td>
</tr>
</tbody>
</table>

Variables are $\mathbb{R}$-valued. For BBS$(1, \infty)$, can understand $(Q_n^t, E_n^t)_{n \in \mathbb{Z}}$ as the lengths of consequence ball/empty box sequences.
**DISCRETE TODA EQUATION (DTODA)**

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<tr>
<td>dToda</td>
<td>$I_{n+1}^t$ $J_{n+1}^t$</td>
<td>$b + c \xrightarrow{ab/b+c} c$</td>
</tr>
<tr>
<td></td>
<td>$U_n^t \xrightarrow{J_n^t, I_{n+1}^t} U_{n+1}^t$</td>
<td>$c \xrightarrow{b \mapsto a, b+c}$</td>
</tr>
</tbody>
</table>

Variables are $(0, \infty)$-valued. UDTODA is obtained as ultra-discrete/ zero-temperature limit by making change of variables:

$$a = e^{-a/\epsilon}, \quad b = e^{-b/\epsilon}, \quad c = e^{-b/\epsilon}.$$
NB. [Quastel, Remenik 2019] connected the KPZ fixed point to the Kadomtsev-Petviashvili (KP) equation. Both dKdV and dToda can be obtained from the discrete KP equation.
2. GLOBAL SOLUTIONS
BASED ON PATH ENCODINGS
Let $\eta$ be a finite configuration.

Define $(S_n)_{n \in \mathbb{Z}}$ by $S_0 = 0$ and

$$S_n - S_{n-1} = 1 - 2\eta_n.$$ 

Let

$$U_n = M_n - S_n,$$

where $M_n = \max_{m \leq n} S_m$.

Can check $(U_n)_{n \in \mathbb{Z}}$ is a carrier process, and the path encoding of $T\eta$ is

$$TS_n = 2M_n - S_n - 2M_0.$$
PITMAN’S TRANSFORMATION

The transformation

\[ S \mapsto 2M - S \]

is well-known as Pitman’s transformation. (It transforms one-sided Brownian motion to a Bessel process [Pitman 1975].)

Given the relationship between \( \eta \) and \( S \), and \( U = M - S \), the relation \( TS = 2M - S - 2M_0 \) is equivalent to:

\[ (T\eta)_n + U_n = \eta_n + U_{n-1}, \]

i.e. conservation of mass.
### ‘PAST MAXIMUM’ OPERATORS

<table>
<thead>
<tr>
<th>Model</th>
<th>‘Past maximum’</th>
<th>Path encoding dynamics</th>
</tr>
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<tbody>
<tr>
<td>udKdV</td>
<td>( M^\nabla(S)<em>n := \sup</em>{m \leq n} \left( \frac{S_m + S_{m-1}}{2} \right) )</td>
<td>( T^\nabla(S) = 2M^\nabla(S) - S )</td>
</tr>
<tr>
<td>dKdV</td>
<td>( M^\Sigma(S)<em>n := \log \left( \sum</em>{m \leq n} \exp \left( \frac{S_m + S_{m-1}}{2} \right) \right) )</td>
<td>( T^\Sigma(S) = 2M^\Sigma(S) - S )</td>
</tr>
<tr>
<td>udToda</td>
<td>( M^{\nabla*}(S)<em>n := \begin{cases} \sup</em>{m \leq \frac{n-1}{2}} S_{2m}, &amp; n \text{ odd}, \ \frac{M^{\nabla*}(S)<em>{n+1} + M^{\nabla*}(S)</em>{n-1}}{2}, &amp; n \text{ even} \end{cases} )</td>
<td>( \mathcal{T}^{\nabla*}(S) = \theta \circ T^\nabla(S) ), where ( T^\nabla(S) := 2M^\nabla(S) - S )</td>
</tr>
<tr>
<td>dToda</td>
<td>( M^{\Sigma*}(S)<em>n := \begin{cases} \log \left( \sum</em>{m \leq \frac{n-1}{2}} \exp \left( S_{2m} \right) \right), &amp; n \text{ odd}, \ \frac{M^{\Sigma*}(S)<em>{n+1} + M^{\Sigma*}(S)</em>{n-1}}{2}, &amp; n \text{ even} \end{cases} )</td>
<td>( \mathcal{T}^{\Sigma*}(S) = \theta \circ T^\Sigma(S) ), where ( T^\Sigma(S) := 2M^\Sigma(S) - S )</td>
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Above corresponds to udKdV\((J, \infty)\) and dKdV\((\alpha,0)\); parameters appear in path encoding. More novel ‘past maximum’ operators for udKdV\((J,K)\), \( J \leq K \) [C., Sasada]. Spatial shift \( \theta \) needed for Toda systems.
‘PAST MAXIMUM’ OPERATORS

\[ T^\vee = \text{udKdV}, \quad T^\sum = \text{dKdV}, \quad T^{\vee*} = \text{udToda}, \quad T^{\sum*} = \text{dToda}. \]
GENERAL APPROACH

Aim to change variables $a_t^n := A_n(\eta_t^n)$, $b_t^n := B_n(u_t^n)$ so that $(a_{t+1}^n, b_t^n) = K_n(a_t^n, b_{t-1}^n)$ satisfies

$$K_n^{(1)}(a, b) - 2K_n^{(2)}(a, b) = a - 2b.$$  

Path encoding given by

$$S_n - S_{n-1} = a_n.$$  

Existence of carrier $(b_n)_{n \in \mathbb{Z}}$ equivalent to existence of ‘past maximum’ satisfying

$$M_n = K_n^{(2)}(S_n - S_{n-1}, M_{n-1} - S_{n-1}) + S_n.$$  

Dynamics then given by $S \mapsto T^M S := 2M - S - 2M_0$.

Advantage: $M$ equation can be solved in examples. Moreover, can determine uniquely a choice of $M$ for which the procedure can be iterated. Gives existence and uniqueness of solutions.
APPLICATION TO BBS($J, \infty$)

Given $\eta = (\eta_n)_{n \in \mathbb{Z}} \in \{0, 1, \ldots, J\}^\mathbb{Z}$, let $S$ be the path given by setting $S_0 = 0$ and $S_n - S_{n-1} = J - 2\eta_n$ for $n \in \mathbb{Z}$. If $S$ satisfies

$$\lim_{n \to \infty} \frac{S_n}{n} > 0, \quad \lim_{n \to -\infty} \frac{S_n}{n} > 0$$

then there is a unique solution $(\eta^t_n, U^t_n)_{n,t \in \mathbb{Z}}$ to udKdV that satisfies the initial condition $\eta^0 = \eta$. This solution is given by

$$\eta^t_n := \frac{J - S^t_n + S^t_{n-1}}{2}, \quad U^t_n := M^\vee (S^t)_n - S^t_n + \frac{J}{2}, \quad \forall n, t \in \mathbb{Z},$$

where $S^t := (T^\vee)^t(S)$ for all $t \in \mathbb{Z}$.

[Essentially similar results hold for other systems.]
APPLICATION TO BBS($J, \infty$)

[Simulation with $J = 1$. For configurations, time runs upwards.]
3. INVARIANT MEASURES 
VIA DETAILED BALANCE
APPROACHES TO INVARIANCE

1. Ferrari, Nguyen, Rolla, Wang: *BBS soliton decomposition*.

2. C., Kato, Tsujimoto, Sasada - *Three conditions theorem for BBS* (later generalized). Any two of the three following conditions imply the third:

   \[ \overset{\leftarrow}{\eta} \overset{d}{=} \eta, \quad \overline{U} \overset{d}{=} U, \quad T\eta \overset{d}{=} \eta, \]

   where \( \overset{\leftarrow}{\eta} \) is the reversed configuration, and \( \overline{U} \) is the reversed carrier process given as \( \overset{\leftarrow}{\eta}_n = \eta_{-(n-1)}, \overline{U}_n = U_{-n} \).

3. C., Sasada - *Detailed balance for locally-defined dynamics.*
DETAILED BALANCE
(HOMOGENEOUS CASE)

Consider homogenous lattice system

\[
\begin{array}{c}
\cdots U_{n-1}^t \rightarrow F \rightarrow U_n^t \rightarrow \cdots,
\end{array}
\]

Suppose \( \mu \) is a probability measure such that \( \mu^\mathbb{Z}(\chi^*) = 1 \), where \( \chi^* \) are those configurations for which there exists a unique global solution.

It is then the case that \( \mu^\mathbb{Z} \circ T^{-1} = \mu^\mathbb{Z} \) if and only if there exists a probability measure \( \nu \) such that

\[
(\mu \times \nu) \circ F^{-1} = \mu \times \nu.
\]

Moreover, when this holds, \( U_n^t \sim \nu \) (under \( \mu^\mathbb{Z} \)).
KDV-TYPE EXAMPLES

\textbf{udKdV} Up to trivial measures and technical conditions, i.i.d. invariant measures are either:
\begin{itemize}
  \item shifted, truncated exponential, or;
  \item scaled, shifted, truncated, bipartite geometric.
\end{itemize}
Carrier marginal is of same form.

\textbf{dKdV}(\alpha,0) I.i.d. invariant measures are given by:
\begin{itemize}
  \item $\mu = GIG(\lambda, c\alpha, c)$ with $2 \int \log(x)\mu(dx) < -\log \alpha$.
\end{itemize}
Carrier marginal of form $\nu = IG(\lambda, c)$.
\textit{Duality gives dKdV}(0,\beta) invariant measures.
\textit{NB. GIG=}generalised inverse Gaussian, \textit{IG=}inverse gamma.

\textbf{Remark} Can check ergodicity of the relevant transformations.
CHARACTERISATION THEOREMS

[Kac 1939] If $X$ and $Y$ are independent, then $X + Y$, $X - Y$ are independent if and only if $X$ and $Y$ are normal with a common variance.

[Matsumoto, Yor 1998], [Letac, Wesolowski 2000] If $X > 0$ and $Y > 0$ are independent, then

$$(X + Y)^{-1}, \quad X^{-1} - (X + Y)^{-1}$$

are independent if and only if $X$ has a generalised inverse Gaussian (GIG) distribution and $Y$ has a gamma distribution.

NB. Appears in study of exponential version of Pitman’s transformation, and random infinite continued fractions.
**CONJECTURE**

$dKdV(\alpha, \beta)$ Detailed balance solution:

$$\mu \times \nu = GIG(\lambda, c\alpha, c) \times GIG(\lambda, c\beta, c).$$

**Conjecture** These are only solutions to detailed balance for $F_{dK}^{(\alpha, \beta)}$. In particular, can [Letac, Wesolowski 2000] be generalised to

$$(X, Y) \mapsto \left( \frac{Y(1 + \beta XY)}{1 + \alpha XY}, \frac{X(1 + \alpha XY)}{1 + \beta XY} \right)$$

with $\alpha \beta > 0$?

**Remark** Our result for udKdV solves (up to technicalities) the 'zero temperature' version based on the map:

$$(X, Y) \mapsto (Y - \max\{X + Y - J, 0\} + \max\{X + Y - K, 0\},$$

$$X - \max\{X + Y - K, 0\} + \max\{X + Y - J, 0\}).$$
SPLITTING TODA-TYPE EXAMPLES

Decompose the map $F_{udT}$ into $F_{udT}^*$ and $F_{udT}^{-1}$:

$$
\begin{align*}
F_{udT}^* &\quad \min\{b, c\} & F_{udT}^{-1} &\quad a+b - \min\{b,c\} \\
\downarrow & & \downarrow & \downarrow \\
c &\quad \frac{c-b}{2} - \frac{\min\{b,c\}}{2} & a &\quad a+c - \min\{b,c\}.
\end{align*}
$$

[Can do similarly for $F_{dT}$.] Invariance of $(\tilde{\mu} \times \mu)^\mathbb{Z}$ for udToda can be related to the existence of $(\tilde{\nu}, \nu)$ such that

$$(\mu \times \nu) \circ F_{udT}^{-1} = (\tilde{\mu} \times \tilde{\nu}),$$

NB. This is also equivalent to local invariance of $\tilde{\mu} \times \mu \times \nu$ under $F_{udT}$, cf. Burke’s property, or to

$$(\tilde{\mu} \times \mu \times \nu) \circ (F_{udT}^{(2,3)})^{-1} = (\mu \times \nu).$$
TODA-TYPE EXAMPLES

udToda Up to trivial measures and technical conditions, alternating i.i.d. invariant measures are either:
- shifted exponential, or;
- scaled, shifted geometric.

dToda Alternating i.i.d. invariant measures are given by:
- gamma distributions.

NB. Can completely characterise detailed balance solutions in these cases using classical results:
- \((X, Y) \mapsto (\min\{X, Y\}, X - Y)\) [Ferguson, Crawford 1964-1966];
- \((X, Y) \mapsto (X + Y, X/(X + Y))\) [Lukacs 1955].
Ergodicity is an open question.
LINKS BETWEEN DETAILED BALANCE SOLUTIONS

Discrete KdV ($\alpha, \beta$):
\[
\begin{align*}
\text{GIG}(\lambda, c\alpha, c) \times \\
\text{GIG}(\lambda, c\beta, c)
\end{align*}
\]

\[
\text{Ultra-discretization:} \\
\lambda(\varepsilon) = \lambda \varepsilon \\
c(\varepsilon) = e^{c/\varepsilon} \\
\alpha(\varepsilon) = e^{-J/\varepsilon} \\
\beta(\varepsilon) = e^{-K/\varepsilon}
\]

\[
\text{Ultra-discrete KdV} (J, K): \\
\text{stExp}(\lambda, c, J - c) \times \\
\text{stExp}(\lambda, c, K - c)
\]

\[
\text{Self-convolution:} \\
K = \infty, \\
(\lambda, c - \frac{J}{2}) \leftrightarrow (\lambda_2, c)
\]

Discrete Toda:
\[
\begin{align*}
\text{Gam}(\lambda_1 + \lambda_2, c) \times \\
\text{Gam}(\lambda_1, c) \times \text{Gam}(\lambda_2, c)
\end{align*}
\]

\[
\text{Ultra-discretization:} \\
\lambda_1(\varepsilon) = \lambda_1 \varepsilon \\
\lambda_2(\varepsilon) = \lambda_2 \varepsilon \\
c(\varepsilon) = e^{c/\varepsilon}
\]

\[
\text{Ultra-discrete Toda:} \\
\text{sExp}(\lambda_1 + \lambda_2, c) \times \\
\text{sExp}(\lambda_1, c) \times \text{sExp}(\lambda_2, c)
\]

\[
\text{Self-convolution:} \\
(\lambda, c) \leftrightarrow (\lambda_2, c)
\]
RELATED STOCHASTIC INTEGRABLE SYSTEMS

cf. [CHAUMONT, NOACK 2018]

\( X_{n}^{t+1} \)

\[ U_{n-1}^{t} \xrightarrow{R(\tilde{X}_{n}^{t}, \cdot, \cdot)} U_{n}^{t} \]

\( (\tilde{\mu} \times \mu \times \nu) \circ R^{-1} = \mu \times \nu \)

\[
\begin{array}{cccc}
R_* & d & R_*^{-1} & e \\
\hline 
c = U_{n-1}^{t} & g & f. & (\mu \times \nu) \circ R_*^{-1} = \tilde{\mu} \times \tilde{\nu} \\
b = X_{n}^{t} & a = \tilde{X}_{n}^{t}
\end{array}
\]

- Directed LPP: \( R = F_{udT}^{(2,3)} \).
- Directed polymer (site weights): \( R = F_{dT}^{(2,3)} \).
  Directed polymer (edge weights), higher spin vertex models...