1. (15 points) (a) Work over the complex numbers. Let \( A = \begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix} \). Find the eigenvalues and state their geometric and algebraic multiplicities.

(b) Is the matrix above diagonalizable? Explain your answer.
In either case, write down a similarity transform putting \( A \) in Jordan normal form.

(c) Let \( B \) be a real matrix with characteristic polynomial \((x + 2)^2(x - 3)^2\). What are the possible Jordan normal forms of \( B \)? To avoid repetition, give your answers with eigenvalues sorted from smallest-in-magnitude to largest-in-magnitude, and if two Jordan forms \( J_1 \) and \( J_2 \) happen to be similar matrices, give only one. You may omit 0-entries if you wish.

2. (15 points) (a) Let \( P_n \) be the \( \mathbb{R} \)-vector space of polynomials of degree at most \( n \) with real coefficients. Let \( S = \{ p_1, \ldots, p_{n+1} \} \subseteq P_n \) be a set of \( n+1 \) polynomials, satisfying \( p_i(0) = 0 \) for all \( i \). Either prove \( S \) is linearly dependent, or give an example to show \( S \) may be linearly independent.

(b) Let \( \vec{u} \) and \( \vec{v} \) be elements of a real inner-product space. Suppose that

\[
|\vec{u} + \vec{v}| = |\vec{u}| + |\vec{v}|
\]

Show that \( \vec{u} \) and \( \vec{v} \) are linearly dependent. Name, or otherwise state clearly, any theorems that you use.

3. (15 points) Let \( k \) be a field. Consider the \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

with entries in \( k \).

(a) Let \( z \in k \) satisfy the sector equation: \( bz^2 + (a - d)z - c = 0 \). Show that \( \vec{v} = \begin{pmatrix} 1 \\ z \end{pmatrix} \) is an eigenvector of \( A \) and determine the corresponding eigenvalue.

(b) Now consider the quotient ring \( R = k[\epsilon]/(\epsilon^2) \). All elements of \( R \) may be written uniquely in the form \( x + \epsilon y \) where \( x, y \in k \). Determine all solutions \( z \in R \) to the equation

\[
\epsilon z^2 - z - 1 = 0.
\]

(c) Let the ring \( R \) be as in the previous part. Find a vector \( \begin{pmatrix} r \\ s \end{pmatrix} \) of elements in \( R \) such that the ideal generated by \( \{r, s\} \) is all of \( R \) and such that

\[
\begin{pmatrix} 1 & \epsilon \\ 1 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \lambda \begin{pmatrix} r \\ s \end{pmatrix}
\]

for some \( \lambda \in R \). Hint: try solving an equation like the sector equation from part (a).
4. (15 points) Let $R$ be a commutative ring and let $f : R \to R$ be a surjective ring homomorphism.

(a) Let $f^n$ denote the composite of $f$ with itself $n$ times. Suppose there exists some integer $m \geq 1$ such that $\ker(f^{m+1}) \subset \ker(f^m)$. Prove that $f$ is injective.

(b) Give an example of a ring $R$ and a homomorphism $f : R \to R$ that is surjective but not injective (you do not have to provide proof).

5. (15 points) Let $p$ be a prime number. Let $k$ be a field of characteristic $p$ and $\overline{k}$ be an algebraic closure of $k$. Let $c \in k$ be an element. Consider the polynomial $f(x) = x^p - x + c$.

a. Suppose $\alpha \in \overline{k}$ is a root $f(\alpha) = 0$. Determine $f(\alpha + 1)$.

b. Prove that if $f$ does not split over $k$, then $f$ is irreducible over $k$.

6. (15 points) Let $D = D_7$ denote the dihedral group of order 14. This group has a presentation $D = \{a, b | a^2 = b^7 = abab = e\}$.

(a) Let $i \in \{0, \ldots, 6\}$. Determine the order of the element $ab^i$ in $D$, in terms of $i$.

(b) Write down all elements of order 7 in $D$.

(c) Consider the set $G$ of all group automorphisms $\phi : D \to D$. The set $G$ forms a group under composition. What is the order of $G$?

(d) Describe a homomorphism $\phi : D \to D$ such that $\phi \neq \text{id}_D$ but such that $\phi \circ \phi \circ \phi = \text{id}_D$. 