Essays in harmonic analysis on p-adic SL(2)

Geometry of the tree

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Suppose $\mathfrak{k}$ to be a $\mathfrak{p}$-adic field—a finite extension of $\mathbb{Q}_p$ or $\mathbb{Z}_p((T))$.

Among many parallels between the structures of real and p-adic groups, one of the most remarkable is that there exist analogues of real symmetric spaces for semi-simple $\mathfrak{p}$-adic groups. These are the **buildings** of Bruhat and Tits. Among them is the tree on which $\mathrm{PGL}_2(\mathfrak{k})$ acts, and that is what this essay and others related to it are all about.

The group $\mathrm{GL}_2(\mathbb{R})$ acts on the space of all symmetric 2×2 real matrices:

$$X\colon\ S \longmapsto X \cdot S \cdot {}^tX\,,$$

and preserves the open cone $\mathcal{C}$ of positive definite matrices. The quotient $\mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\{\text{scalars}\}$ therefore acts on the space $\mathbb{P}(\mathcal{C})$, the space of such matrices modulo positive scalars. In effect, $\mathbb{P}(\mathcal{C})$ parametrizes the shapes of ellipses in the plane. The isotropy subgroup of I is the image $\overline{\mathrm{O}}_2 = \mathrm{O}_2/\{\pm I\}$ in $\mathrm{PGL}_2(\mathbb{R})$, so that $\mathbb{P}(\mathcal{C})$ may be identified with $\mathrm{PGL}_2(\mathbb{R})/\overline{\mathrm{O}}_2$. The embedding of SL_2 into GL_2 identifies this with $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$.

The hollow cone $\mathcal{C}_0$ of non-negative symmetric matrices of rank one that borders $\mathcal{C}$ is also stable under $\mathrm{GL}_2(\mathbb{R})$. To each point of $\mathcal{C}_0$ corresponds the null line of the corresponding quadratic form, and $\mathbb{P}(\mathcal{C}_0)$ may be identified with $\mathbb{P}^1(\mathbb{R})$, the space of lines in $\mathbb{R}^2$. This space compactifies $\mathbb{P}(\mathcal{C})$.

If we choose coordinates

$$\begin{bmatrix} z+y & x \\ x & z-y \end{bmatrix}$$

for symmetric matrices, the space $\mathcal{C}$ is where z, $z^2 - y^2 - x^2 > 0$. The intersection of this and the plane $z = 1$ is the open disc $x^2 + y^2 < 1$, which may be identified with $\mathbb{P}(\mathcal{C})$.

There exists a Riemannian metric on $\mathbb{P}(\mathcal{C})$, invariant with respect to $\mathrm{PGL}_2(\mathbb{R})$ and unique with this property, up to a positive scalar multiple. In the obvious scheme, geodesics are straight line segments inside the unit disk—this is the Klein model. The Poincaré model is derived from this by stereographic projection, and in this model geodesics are circular arcs intersecting the unit circle at right angles. In the Poincaré model, $\mathrm{SL}_2(\mathbb{R})$ acts by fractional linear transformations. Interesting representations of $\mathrm{SL}_2(\mathbb{R})$ are obtained on eigenspaces of the non-Euclidean Laplacian.

The Bruhat-Tits tree is the analogue of this for $\mathfrak{p}$-adic fields. It parametrizes norms on $\mathfrak{k}^2$ of a certain type, modulo similarity. It is a graph whose nodes are in bijection with $\mathrm{PGL}_2(\mathfrak{k})/K$, where now $K = \mathrm{PGL}_2(\mathfrak{o})$. In this essay I shall define it and prove some of its elementary properties, and in later essays show how it can be used in harmonic analysis on $\mathrm{SL}_2(\mathfrak{k})$ and $\mathrm{PGL}_2(\mathfrak{k})$. Very little of what I'll say is original, but the material is widely scattered in the literature, and sometimes available in only a sketchy manner.

The space $\mathbb{P}(\mathcal{C})$ is the simplest of the non-compact symmetric spaces. In general, there corresponds one of these to every semi-simple real Lie group G. It is isomorphic to G/K, where K is a maximal compact subgroup of G, and parametrizes certain involutions of the Lie algebra of G. For groups of higher rank, buildings generalize the trees constructed here. They are important in understanding the structure of such groups, but play a relatively small role in analysis. Doing analysis on the tree of $\mathrm{SL}_2(\mathfrak{k})$ offers a unique opportunity to understand many analytical phenomena intuitively.

The standard reference for the material in this part is Chapitre II of [Serre:1977].

This is one of a collection of essays on different topics all related in some way to harmonic analysis on $\mathrm{SL}_2(\mathfrak{k})$, with $\mathfrak{k}$ a $\mathfrak{p}$-adic field. It began as a set of notes about applications of the Bruhat-Tits tree associated to $\mathrm{SL}_2(\mathfrak{k})$, but has grown enormously.

Much of this essay was written while I was giving a series of lectures on related material at the Tata Institute for Fundamental Research. I wish to thank the Institute for its hospitality, and my audience for their patience.

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1. Notation

Throughout the entire collection of essays on the tree of Bruhat-Tits, let

$\mathfrak{k} =$ a field with a discrete valuation
$\mathfrak{o} =$ the associated ring of integers
$\mathfrak{p} =$ the maximal ideal of $\mathfrak{o}$
$\varpi =$ a generator of $\mathfrak{p}$
$q =$ the cardinality of $\mathfrak{o}/\mathfrak{p}$, assumed to be finite

The quotient $\mathfrak{o}/\mathfrak{p}$ is isomorphic to the Galois field $\mathbb{F}_q$. Every $x \neq 0$ in $\mathfrak{k}$ can be factored as $u\varpi^k$ with u a unit in $\mathfrak{o}$. The norm

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ q^{-k} & \quad x = u\varpi^k \end{cases}$$

on $\mathfrak{k}$ satisfies the non-Archimedean conditions

$$\begin{aligned} |xy| &= |x|\,|y| \\ |x+y| &= \max |x|, |y| \quad \text{if } |x| \neq |y| \\ &\leq |x| = |y| \qquad \text{otherwise.} \end{aligned}$$

The ring $\mathfrak{o}$ is the set of x with $|x| \leq 1$, and there exists on $\mathfrak{k}$ a unique Haar measure assigning $\mathfrak{o}$ measure 1. Multiplication by $x \neq 0$ scales measures by $|x|$.

I shall write $*\varpi^n$ for a general element of $\mathfrak{k}$ of norm q^{-n}.

Eventually, $\mathfrak{k}$ will be required to be complete. This means that it is complete in the topology defined by the norm, and this happens if and only if $\mathfrak{o}$ is the projective limit of quotients $\mathfrak{o}/\mathfrak{p}^n$. In this case $\mathfrak{k}$ is locally compact. One useful example in which completeness does not hold is that in which $\mathfrak{k} = \mathbb{Q}$ and $\mathfrak{o}$ is the ring $\mathbb{Z}_{(p)}$, the ring of rational numbers a/b with b relatively prime to the prime number p. One point of the less restrictive assumption is that one might want to implement algorithmically some of the results presented here.

Although I shall be working with a limited range of reductive groups—the groups SL_2, GL_2, PGL_2 and tori contained in them—it will be convenient to use some notation and terminology from the general theory.

ALGEBRAIC TORI. An **algebraic torus** defined over a base field is an algebraic group that becomes isomorphic to a product of multiplicative groups upon base field extension. Of course, one example is itself a product of multiplicative groups, in which case it is said to be **split**.

If T is an algebraic torus, let $X^*(T)$ be the lattice of algebraic homomorphisms from T to the multiplicative group $\mathbb{G}_m$ defined over $\mathfrak{k}_s$. The dual lattice is $X_*(T)$, that of homomorphisms from the multiplicative group into T. If T is the group of diagonal matrices in SL_2, for example, this last is the free module of rank one spanned by the map

$$x \longmapsto \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}.$$

The duality of $X^*(T)$ and $X_*(T)$ is expressed by the equation

$$\lambda(\mu^\vee(x)) = x^{\langle \lambda, \mu^\vee \rangle}.$$

If T is not split over $\mathfrak{k}$ it is completely characterized by the action of the Galois group on $X_*(T)$, and the maximal split torus A in T is that whose group $X_*(A)$ is the sublattice whose elements are fixed by the Galois group. I shall often use additive notation for $X_*(T)$, so that $\lambda^\vee$ maps x to $x^{\lambda^\vee}$.

There will be two types of non-trivial tori of principal interest. Suppose $E/\mathfrak{k}$ to be a separable quadratic extension, generated by a root ζ of the quadratic equation

$$x^2 - bx + c = 0 = (x-\zeta)(x-\overline{\zeta})\,.$$

The norm of $x + y\zeta$ is

$$(x+y\zeta)(x+y\overline{\zeta}) = x^2 + bxy + cy^2\,.$$

One of the two associated tori is the multiplicative group of E, considered as an algebraic torus defined over $\mathfrak{k}$. This is the algebraic variety (of dimension two)

$$\big\{(x,y,z)\,\big|\,(x^2+bxy+cy^2)z = 1\big\}\,.$$

The multiplicative group of $\mathfrak{k}$ itself sits in here as the diagonal subgroup. The other torus is the group $N^1_{E/\mathfrak{k}}$ of elements of E of norm 1, the variety

$$x^2 + bxy + cy^2 = 1\,.$$

The group of characters of the first is generated by

$$(x,y,z) \longmapsto x+y\zeta,\ x+y\overline{\zeta}$$

and conjugation swaps them. The character group of the second is generated by either $x+\zeta y$ or $x-\zeta y$. Conjugation swaps these, and hence acts as multiplication by -1 on its character group.

REDUCTIVE GROUPS. If G is one of the three groups SL_2 or GL_2, then

$$\begin{aligned}
P &= \text{the subgroup of upper triangular matrices in } G\\
A &= \text{the subgroup of diagonal matrices in } P\\
N &= \text{the subgroup of unipotent matrices in } P\\
\overline{N} &= \text{the group opposite to } N\\
\delta &= \text{the character of the adjoint action of } A \text{ on the Lie algebra } \mathfrak{n}\\
K &= G(\mathfrak{o})\\
w &= \begin{bmatrix} \circ & -1 \\ 1 & \circ \end{bmatrix}\,.
\end{aligned}$$

I shall add a subscript if the context requires a distinction among the groups G—for example A_{GL} or A_{SL}.

If H is any algebraic subgroup of G, I'll let $H(\mathfrak{o})$ be the matrices in H with entries in $\mathfrak{o}$, and $H(\mathfrak{p}^m)$ the subgroup of h in $H(\mathfrak{o})$ with $h \equiv I \bmod \mathfrak{p}^m$.

The valuation on $\mathfrak{k}$ allows one to identify $X_*(A)$ with $A/A(\mathfrak{o})$, taking $\lambda^\vee$ to $\lambda^\vee(\varpi)$. This allows us to identify $X_*(A)$ with the particular subgroup

$$\boldsymbol{A} = \big\{\lambda^\vee(\varpi)\,\big|\,\lambda^\vee \in X_*(A)\big\}$$

generated by the image of $X_*(A)$. The character

$$\delta\colon\ a \longmapsto |\det \mathrm{Ad}_{\mathfrak{n}}(a)|$$

makes sense for all these groups. In each case, define

$$\begin{aligned}
A^{--} &= \{a \in A \,|\, \delta(a) \le 1\}\\
A^{++} &= \{a \in A \,|\, \delta(a) \ge 1\}\\
\boldsymbol{A}^{--} &= \text{ intersection of } \boldsymbol{A} \text{ with } A^{--}\\
\boldsymbol{A}^{++} &= \text{ intersection of } \boldsymbol{A} \text{ with } A^{++}
\end{aligned}$$

For each of the groups $G = \mathrm{GL}_2$ or SL_2, I define α_G to be a particular element of $A/A(\mathfrak{o})$. Thus

$$\begin{aligned}
\alpha_{\mathrm{GL}} = \alpha &= \begin{bmatrix} \varpi & \circ \\ \circ & 1 \end{bmatrix} \text{ modulo scalars}\\
\alpha_{\mathrm{SL}} &= \begin{bmatrix} \varpi & \circ \\ \circ & 1/\varpi \end{bmatrix}\,.
\end{aligned}$$

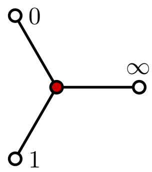
2. Lines

The tree of SL_2 is an elaboration of a much more elementary object. In this section, begin by supposing k to be an arbitrary field.

The projective line $\mathbb{P} = \mathbb{P}^1(k)$ is by definition the set of lines through the origin in k^2. To it is associated a graph. The nodes in the graph are all the non-zero subspaces in k^2, and two of these are connected by an edge if and only if one is contained in the other. There are thus two types of nodes, a central node representing k^2 itself, and one for each point of $\mathbb{P}$. The only edges in this graph are those from the center to the node of a line.

The points of $\mathbb{P}$ are parametrized by non-zero points of k^2 modulo scalar multiplication. The set $\mathbb{P}$ contains a copy of k itself, the line $((x, 1))$ through $(x, 1)$. In addition there is the line $((1, 0))$. By convention, the line $((x, y))$ is labeled by x/y, which makes the last point ∞. There is a second notable copy of k in $\mathbb{P}$ as well, the points $((1, x))$. Each of these copies is an affine algebraic variety, and the coordinate transformation on their intersection is $x \mapsto 1/x$ since $((x, 1)) = ((1, 1/x))$. This defines $\mathbb{P}$ as an algebraic variety over k.

If $k = \mathbb{F}_q$ there are $q + 1$ points in $\mathbb{P}$. If $k = \mathbb{F}_2$, for example, the graph looks like this:



The group $\mathrm{GL}_2(k)$ acts on k^2, and this induces a transitive action of $\mathrm{PGL}_2(k)$ on $\mathbb{P}$. The stabilizer of ∞ is the group P of upper triangular matrices. The stabilizer of 0 is the opposite parabolic subgroup $\overline{P}$ of lower triangular matrices. The covering of $\mathbb{P}$ by two copies of k corresponds to the covering of GL_2 by NwP and $\overline{N}P$. Explicitly:

2.1. Lemma. (Bruhat factorization) *Let*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

(a) The matrix g belongs to $NwP = PwN$ if and only if $c \neq 0$, and then (with $\Delta = \det$)

$$\begin{aligned} g &= \begin{bmatrix} 1 & a/c \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \circ & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & \circ \\ \circ & \Delta/c \end{bmatrix} \begin{bmatrix} 1 & d/c \\ \circ & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a/c \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \Delta/c & \circ \\ \circ & c \end{bmatrix} \begin{bmatrix} \circ & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ \circ & 1 \end{bmatrix} . \end{aligned}$$

(b) The matrix g belongs to $\overline{N}P$ if and only if $a \neq 0$, and then

$$g = \begin{bmatrix} 1 & \circ \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & \circ \\ \circ & \Delta/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ \circ & 1 \end{bmatrix} .$$

Now suppose k to be a local field $\mathfrak{k}$. A point (x, y) in $\mathfrak{o}^2$ is called **primitive** if one of x and y is a unit.

2.2. Lemma. *The pair (a, c) is primitive if and only if the $\mathfrak{o}$-module generated by (a, c) is a summand of $\mathfrak{o}^2$.*

Proof. One can easily find a matrix

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

whose determinant is a unit. Therefore its columns make a basis of $\mathfrak{o}^2$.

The space $\mathbb{P}(\mathfrak{o})$ is by definition the set of all $\mathfrak{o}$-submodules of $\mathfrak{o}^2$ that are summands—the lines of $\mathfrak{o}^2$. According to the Lemma, this set may be identified with the set of primitive pairs modulo scalar multiplication by units. If ℓ is a line in $\mathbb{P}(\mathfrak{k})$ then $\ell \cap \mathfrak{o}^2$ is a line in $\mathfrak{o}^2$, and we thus get a canonical map from $\mathbb{P}(\mathfrak{k})$ to $\mathbb{P}(\mathfrak{o})$. The following is immediate:

2.3. Lemma. *This map is a bijection.*

For u generating a line in $\mathfrak{k}^2$ let $[\![u]\!]$ be the corresponding $\mathfrak{o}$-module, and $\langle\!\langle u \rangle\!\rangle$ the corresponding line in $\mathbb{P}(\mathfrak{k})$. If we are given a basis (u, v) of $\mathfrak{o}^2$ and x is in $\mathfrak{k} \cup \{\infty\}$, I'll write x instead of $v + xu$: thus $[\![x]\!]$ and $\langle\!\langle x \rangle\!\rangle$.

3. Lattices

A **lattice** in $\mathfrak{k}^2$ is any finitely generated $\mathfrak{o}$-submodule that spans $\mathfrak{k}^2$ as a vector space, for example $\mathfrak{o}^2$.

3.1. Proposition. *Every lattice in $\mathfrak{k}^2$ is free over $\mathfrak{o}$ of rank 2.*

Proof. The proof will be constructive. The lattice is finitely generated, so we may suppose given m generators of the $\mathfrak{o}$-submodule L, and suppose $M = M_L$ to be the $2 \times m$ matrix whose columns are those generators. Multiplying by matrices in $\mathrm{GL}_n(\mathfrak{o})$ on the right does not change the lattice the columns generate. The Proposition is therefore a consequence of:

3.2. Lemma. *Every $2 \times n$ matrix of rank 2 with entries in $\mathfrak{k}$ may be reduced through multiplication on the right by a matrix in $\mathrm{GL}_n(\mathfrak{o})$ to one whose non-zero columns are of the form*

$$\begin{bmatrix} \varpi^m & x \\ \circ & \varpi^n \end{bmatrix}.$$

The integers m, n are unique, and the entry x is unique modulo $\mathfrak{p}^m$.

Proof of the Lemma. I'll specify precisely what multiplications need to be carried out. These will be what I call **integral column operations**.

There are three types of integral column (or, for that matter, row) operations:

(a) permuting columns (rows);
(b) multiplying one column (row) by a unit of $\mathfrak{o}$;
(c) adding to any column (row) an integral multiple of another.

Thes column operations may be effected through multiplication on the right by matrices in $\mathrm{GL}_m(\mathfrak{o})$, typically embedded copies of

$$\begin{bmatrix} \circ & 1 \\ 1 & \circ \end{bmatrix}, \quad \begin{bmatrix} u & \circ \\ \circ & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \circ \\ \circ & u \end{bmatrix}, \quad \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \circ \\ x & 1 \end{bmatrix}.$$

Multiplication on the right by any one of these clearly does not change the lattice generated by the columns.

Row operations can be carried out through multiplication on the left by matrices in $\mathrm{GL}_2(\mathfrak{o})$, and amount to a change of basis in $\mathfrak{k}^2$.

Now to start the description of the process. Because the lattice has rank 2, there exists at least one non-zero entry in the second row. One among them will have maximal norm, and we may swap columns if necessary to get it into the lower left corner. By an operation of type (b), we may make it ϖ^n for some n, and then we may apply operations of type (c) to reduce the rest of the second row to 0.

We now look at the first row. Again beginning with a swap if necessary, possibly followed by a unit column multiplication, we may get an entry in position $(1, 2)$ of the form ϖ^m and of maximal norm in columns $c \geq 2$. We may then apply operations of type (c) to make the first row in columns 3 to n vanish. The only non-zero entries are now in columns $1, 2$, giving us this:

$$\begin{bmatrix} x & \varpi^m \\ \varpi^n & \circ \end{bmatrix}.$$

A column swap will conclude the computation.

As for uniqueness, let $\mathfrak{o}_0$ be the module generated by $(1, 0)$. Then $(\varpi^m, 0)$ is a generator of $L \cap \mathfrak{o}_0$ and ϖ^{m+n} determined by $\det(g)$.

Let $G = \mathrm{GL}_2(\mathfrak{k})$ or $\mathrm{SL}_2(\mathfrak{k})$, with other notation as in the Introduction. I recall that $K = G(\mathfrak{o})$.

3.3. Corollary. (Iwasawa factorization) *Every matrix in G can be expressed as*

$$\begin{bmatrix} \varpi^m & x \\ \circ & \varpi^n \end{bmatrix} k = \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \varpi^{m-n} & \circ \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \varpi^n & \circ \\ \circ & \varpi^n \end{bmatrix} k$$

with k in K, unique m, n, and x in $\mathfrak{k}$ unique modulo $\mathfrak{p}^m$.

This implies also that any element of $G(\mathfrak{k})$ can be factored as kp with k in K, p in P.

Proof. Only the case of SL_2 is not an immediate consequence of the Lemma. But if g in $\mathrm{SL}_2(\mathfrak{k})$ be expressed in the form of the Corollary, we can replace p by pu and k by $u^{-1}k$, with u in $P \cap \mathrm{GL}_2(\mathfrak{o})$ to obtain k in $\mathrm{SL}_2(\mathfrak{o})$.

Corollary 3.3 is equivalent to the claim that K acts transitively on $\mathbb{P}^1(\mathfrak{k})$.

The group $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on bases of $\mathfrak{k}^2$, hence also on the set of lattices. The stabilizer of $\mathfrak{o}^2$ is K, so with that choice of base lattice the set of lattices may be identified with $\mathrm{GL}_2(\mathfrak{k})/K$.

3.4. Proposition. (Principal divisor theorem) *Given a matrix g in G, there exist matrices k_1, k_2 in K and a diagonal matrix*

$$d = \begin{bmatrix} \varpi^m & \circ \\ \circ & \varpi^n \end{bmatrix}$$

such that

$$g = k_1 d k_2 \, .$$

The diagonal matrix d is unique up to permutation of the diagonal entries.

Proof. The proof is a variation on that of the Lemma 3.2. By column and row permutations, we may assume that the left corner entry is that of maximal norm in the entire matrix, and by a unit column multiplication we may assume it to be ϖ^m. Row and column operations of type (3), followed by a unit column multiplication, make it of the right form.

As for uniqueness, after a swap if necessary, the greatest common divisor of the entries of the matrix is ϖ^m, and $|\det(g)| = |{*}\varpi^{m+n}|$.

3.5. Corollary. *If L and M are two lattices, there exists a basis (e, f) of L and integers $m \le n$ such that $(\varpi^m e, \varpi^n f)$ is a basis of M.*

In these circumstances I call $[\varpi^m : \varpi^n]$ the matrix index of the pair (L, M) and q^{m+n} the index. If $m, n \ge 0$ this last is indeed the index, the size of L/M. If $L = \mathfrak{o}^2$ and (e, f) form an $\mathfrak{o}$-basis of M, this is also

$$\big|\det[\, e \ f \,]\,\big|^{-1} .$$

Proof. I suppose L and M to be given as 2×2 matrices λ and μ of rank 2. Use a coordinate system in which $L = \mathfrak{o}^2$. This means replacing λ by I and μ by $\lambda^{-1}\mu$. Apply the previous Proposition to it. The columns of k_1 form a basis of L, and those of $\mu k_2^{-1} = k_1 d$ form one of M.

As a consequence of the proof of Lemma 3.2:

3.6. Corollary. *The group K is generated by the matrices corresponding to integral column operations.*

In fact, we can be a bit more explicit about this.

3.7. Proposition. *The group K is the disjoint union*

$$N(\mathfrak{o}) w P(\mathfrak{o}) \sqcup \overline{N}(\mathfrak{p}) P(\mathfrak{o}) \, .$$

Proof. This follows from the Bruhat decomposition (Lemma 2.1) of $G(\mathbb{F})$. If the matrix g lies in K and c is a unit, then part (a) implies that g lies in $N(\mathfrak{o})wP(\mathfrak{o})$. If c is not a unit it will lie in $\mathfrak{p}$, and a will be a unit. Then (b) tells us that

$$g = \begin{bmatrix} 1 & \circ \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & \circ \\ \circ & \det/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ \circ & 1 \end{bmatrix} .$$

3.8. Proposition. *Every g in GL_2 may be represented in one of two forms:*

$$g = n \begin{bmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varpi^m & \circ \\ \circ & \varpi^m \end{bmatrix} k \quad (\ell \ge 0) \quad \text{or} \quad g = \overline{n} \begin{bmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{bmatrix} \begin{bmatrix} \varpi^m & \circ \\ \circ & \varpi^m \end{bmatrix} k \quad (\ell > 0) \, .$$

Here n lies in $N(\mathfrak{o})$, $\overline{n}$ in $\overline{N}(\mathfrak{o})$. The integers ℓ, m are unique.

This can also be formulated as saying that the group G is the union of

$$N(\mathfrak{o})\boldsymbol{A}^{--}K \ \text{ and } \ \overline{N}(\mathfrak{p})\boldsymbol{A}^{++}K \, .$$

The intersection of the two pieces is the product of K and the centre of G.

Proof. It will be useful to have explicit formulas. According to Proposition 3.4, let $g = k_1\alpha^{-m}zk_2$ with $m \ge 0$, z a diagonal matrix, and say

$$k_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

If c is a unit, we can write $k_1 = nwan_*$ with

$$n = \begin{bmatrix} 1 & a/c \\ \circ & 1 \end{bmatrix} \ \text{ in } \ N(\mathfrak{o}) \, ,$$

and then, since $\alpha^m P(\mathfrak{o})\alpha^{-m} \subseteq P(\mathfrak{o})$:

$$k_1\alpha^{-m}k_2 = n\alpha^m \cdot \text{something in } KZ_G \ .$$

If c is not a unit we can write $k_1 = \overline{n}an$ with

$$\overline{n} = \begin{bmatrix} 1 & \circ \\ c/a & 1 \end{bmatrix} ,$$

and then

$$k_1\alpha^{-m}k_2 = \overline{n}\alpha^{-m} \cdot \text{something in } KZ_G \ .$$

It will be seen later that this has a simple interpretation in terms of the geometry of the tree.

A subset Ω of $\mathrm{GL}_2(\mathfrak{k})$ **bounded** if the matrix entries of the elements of Ω and Ω^{-1} are bounded. If $\mathfrak{k}$ is complete, Ω is bounded if and only if it is compact. The product of any number of bounded sets is bounded. By Proposition 3.4, a set is bounded if and only if it is contained in the union of a finite number of double cosets $K\varpi^\lambda K$. Any bounded set is contained in the union of a finite number of translates of K.

3.9. Proposition. *The stabilizer of any lattice in $\mathrm{GL}_2(\mathfrak{k})$ is a bounded subgroup. If $\mathfrak{k}$ is complete, it is open and compact. Conversely, any bounded subgroup stabilizes some lattice.*

Proof. For this, one may as well replace $\mathfrak{k}$ by its completion. If $L = g(\mathfrak{o}^2)$ then its stabilizer is gKg^{-1}.

Suppose K_* to be a bounded subgroup, and let $L = \mathfrak{o}^2$. The intersection of two, hence of any finite number of, lattices is again a lattice. If $H = K_* \cap K$, then K_*/H is in bijection with K_*K/K, hence finite. If $K_* = \bigcup k_iH = \bigcup_{k\in K} kH$ then $\bigcap k_iL = \bigcap_{k\in K} kL$ is a lattice stable under it.

4. The tree

The Bruhat-Tits tree of $G = \mathrm{SL}_2(\mathfrak{k})$ is a graph $\mathfrak{X}$ on which the group $\mathrm{PGL}_2(\mathfrak{k})$ acts. The geometry of this graph encodes much of the group structure.

- *The* **nodes** *of the graph are the lattices in $\mathfrak{k}^2$ modulo similarity.*

For each lattice L let $\langle\!\langle L\rangle\!\rangle$ be the corresponding node of the graph, or in other words its equivalence class, the set of lattices $\{\varpi^n L\}$.

If L and M are lattices, Corollary 3.5 asserts that we can find a basis (e, f) of L such that $(\varpi^m e, \varpi^n f)$ is a basis of M, for some integers $m \le n$. The difference $n - m$ is an invariant of the similarity class of M, so that the definition $\mathrm{inv}(\langle\!\langle L\rangle\!\rangle \colon \langle\!\langle M\rangle\!\rangle) = n - m$ makes sense. This invariant is 1 if and only if the two nodes possess representatives L and M with $L/M \cong \mathfrak{o}/\mathfrak{p}$, or equivalently

$$\varpi L \subset M \subset L\,.$$

In this case, I'll call them **neighbours**. Being neighbours is a symmetric relationship.

- *There is an* **edge** *of the Bruhat-Tits graph between two nodes if and only if they are neighbours.*

The nodes linked by an edge to $\langle\!\langle L\rangle\!\rangle$ thus correspond to lines of $L/\varpi L \cong (\mathbb{F}_q)^2$, and there are $q + 1$ of them.

If u and v form a basis of $\mathfrak{k}^2$, let $[\![u, v]\!]$ be the lattice they span and $\langle\!\langle u, v\rangle\!\rangle$ the corresponding node. (As we shall see later, this is consistent with my earlier notation for lines in $\mathfrak{k}^2$ and $\mathfrak{o}^2$.)

The space $\mathbb{P}^1(\mathbb{F}_q)$ may be identified with the union of a copy of $\mathbb{F}_q$ and a point called ∞. The first are the lines in $\mathbb{F}^2$ through the points $(x, 1)$, and the second the single line through $(1, 0)$. This implies:

- *The neighbours of $\langle\!\langle u, v\rangle\!\rangle$ are the $\langle\!\langle \varpi u, xu + v\rangle\!\rangle$ as x ranges over $\mathfrak{o}/\mathfrak{p}$, together with $\langle\!\langle u, \varpi v\rangle\!\rangle$.*

Fix once and for all basis vectors and particular nodes

$$\begin{aligned} u_0 &= (1, 0) \\ v_0 &= (0, 1) \\ \nu_m &= \langle\!\langle \varpi^m u_0, v_0\rangle\!\rangle = \langle\!\langle u_0, \varpi^{-m} v_0\rangle\!\rangle \quad (m \in \mathbb{Z})\,, \end{aligned}$$

so that ν_0 is the equivalence class of $\mathfrak{o}^2$. This choice makes possible the notation $[\![x]\!]$ and $\langle\!\langle x\rangle\!\rangle$ for $[\![v_0 + xu_0]\!]$ and $\langle\!\langle v_0 + xu_0\rangle\!\rangle$.

Any g in $\mathrm{GL}_2(\mathfrak{k})$ takes a lattice $[\![u, v]\!]$ to the lattice $[\![gu, gv]\!]$. The group $\mathrm{GL}_2(\mathfrak{k})$ preserves equivalence of lattices, and it also preserves the lattice pair invariant. Hence it transforms edges to edges, and therefore acts as an automorphism of the graph $\mathfrak{X}$. By definition, this action factors through $\mathrm{PGL}_2(\mathfrak{k})$. The group $\mathrm{PGL}_2(\mathfrak{k})$ acts transitively on nodes of the graph. The stabilizer in $\mathrm{PGL}_2(\mathfrak{k})$ of the node ν_0 is the maximal compact subgroup $\mathrm{PGL}_2(\mathfrak{o})$, which is therefore the analogue in $\mathrm{PGL}_2(\mathfrak{k})$ of the image of $\mathrm{O}(2)$ in $\mathrm{PGL}_2(\mathbb{R})$.

Suppose that $L = \mathfrak{o}^2$ and that the matrix index of $[L \colon M]$ is $[\varpi^m \colon \varpi^n]$. I call $\langle\!\langle M\rangle\!\rangle$ **even** or **odd** depending on the parity of $n - m$. The action of $\mathrm{SL}_2(\mathfrak{k})$ preserves this parity, and in fact there are exactly two orbits of the group $\mathrm{SL}_2(\mathfrak{k})$ among the nodes of the graph, each one corresponding to lattices of a given parity.

A **chain** of lattices is a finite or half-infinite sequence of lattices

$$L_0 \supset L_1 \supset \ldots \supset L_n \supset L_{n+1} \ldots$$

with

$$L_n \supset L_{n+1} \supset \varpi L_n$$

for all n. A chain of nodes is the sequence of nodes associated to such a sequence of lattices.

A **standard chain** of lattices is one of the form

$$[\![u_0, v_0]\!] — [\![\varpi u_0, v_0]\!] — [\![\varpi^2 u_0, v_0]\!] — \cdots ,$$

whether finite or infinite. I'll call a chain **simple** if, like this one, it does not back-track—i.e. none of the L_i are equivalent.

4.1. Proposition. *Every finite simple chain of lattices may be transformed to a standard one by an element of* $\mathrm{GL}_2(\mathfrak{k})$*. If* $\mathfrak{k}$ *is complete, this is true of any chain.*

Proof. The proof is by induction on the length of the chain

$$L_0 \supset L_1 \supset \cdots \supset L_n ,$$

in which we may assume $L_k \supset L_{k+1} \supset \varpi L_k$ for all k. It will be constructive.

Since $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on nodes, we may assume that $L_0 = \mathfrak{o}^2$.

If $n = 1$, the image of L_1 in $L_0/\varpi L_0$ is a line. We can find a matrix $\overline{g}$ in $\mathrm{GL}_2(\mathbb{F}_q)$ transforming it to the line through $(1, 0)$, and if g in $\mathrm{GL}_2(\mathfrak{o})$ has image $\overline{g}$, then gL_1 is $[\![\varpi, 1]\!]$, corresponding to the node ν_1.

The first part of the Proposition will now follow from this:

4.2. Proposition. *Any finite simple chain that starts out* $[\![u_0, v_0]\!] — [\![\varpi u_0, v_0]\!]$ *may be transformed to a standard chain with the same initial pair by an element of the form*

$$\begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}$$

with x *in* $\mathfrak{p}$.

This will follow by induction from:

4.3. Lemma. *Suppose given a finite chain* (L_i) $(0 \le i \le n+1)$ *with* $L_i = [\![\varpi^i, 1]\!]$ *for* $1 \le i \le n$*. There exists* $x \in \mathfrak{p}^n$ *such that*

$$\begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix}$$

takes every L_i *to* $[\![\varpi^i, 1]\!]$.

Implicit in this statement is that when x lies in $\mathfrak{p}^n$ this matrix takes $[\![\varpi^i, 1]\!]$ to itself for every $i \le n$.

Proof of the Lemma. We know that the lattices linked to ν_0 are $\langle\!\langle u_0, \varpi v_0\rangle\!\rangle$ and the $\langle\!\langle \varpi u_0, x u_0 + v_0\rangle\!\rangle$ as x ranges over $\mathfrak{o}/\mathfrak{p}$. Translating by α^m, we see that the lattices linked to ν_m are $\langle\!\langle \varpi^{m-1} u_0, v_0\rangle\!\rangle$ and the $\langle\!\langle \varpi^{m+1} u_0, x u_0 + v_0\rangle\!\rangle$ as x ranges over $\mathfrak{p}^m/\mathfrak{p}^{m+1}$. The first one amounts to a back-track. For the last set, the Lemma is clear. ▯

To conclude the proof of Proposition 4.1: when $\mathfrak{k}$ is complete, the product of the matrices

$$\begin{bmatrix} 1 & x_n \\ \circ & 1 \end{bmatrix}$$

found inductively will then converge. ▯

4.4. Corollary. *Every finite simple chain of nodes in the graph may be transformed to a standard one by an element of* $\mathrm{GL}_2(\mathfrak{k})$*. If* $\mathfrak{k}$ *is complete, this is true of any chain.*

4.5. Corollary. *The distance* $|x\!:\!y|$ *between two nodes* x *and* y *is the pair invariant* $\mathrm{inv}(x\!:y)$.

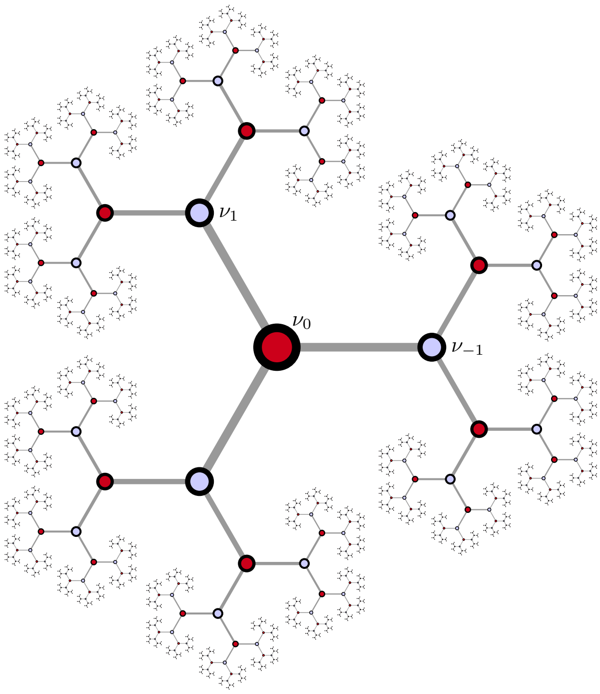
Only a short additional argument is necessary to prove:

4.6. Corollary. *The graph $\mathfrak{X}$ is a connected tree.*

Proof. If M is any lattice, we may find a basis (e,f) of $L=\mathfrak{o}^2$ such that some $(\varpi^m e, \varpi^n f)$ is a basis of M. Replacing M by some multiple of itself, then we may assume $m=0$, $n\geq 0$. That means that there exists a chain of lattices $[[u_0, \varpi^k v_0]]$ from L to M. This proves that the graph is connected. That it is a tree follows from the preceding Proposition, since no standard chain has a loop. ∎

The node ν_0 may be chosen as root.

The structure of $\mathfrak{X}$ is completely determined by the properties: (a) it is connected; (b) it is a tree; (c) every node has $q+1$ neighbours. For example, when $q=2$ it looks like this:



5. Compactification

The tree for $\mathfrak{k}$ and its completion are the same. What is different between the two is a certain asymptotic behaviour at infinity.

- *From now on, I shall assume $\mathfrak{k}$ to be complete.*

5.1. Proposition. *Suppose*

$$L_0 \supset L_1 \supset L_2 \supset \ldots$$

to be a simple chain of lattices in $\mathfrak{k}^2$. Then

$$\bigcap L_n = \ell \cap \mathfrak{o}^2$$

for some unique line ℓ in $\mathfrak{k}^2$.

That is to say that in some sense the lattices L_n have ℓ as limit.

Proof. From Proposition 4.1, since the claim is true if $L_n = [[\varpi^n u_0, v_0]]$.

The tree possesses a rather simple and obvious topology. The previous result suggests how to compactify it by adding to it the points of $\mathbb{P}^1(\mathfrak{k})$. Therefore I define $\overline{\mathfrak{X}}$ to be the union of $\mathfrak{X}$ and $\mathbb{P}(\mathfrak{k})$. It remains to define on it a topology. The topologies on $\mathfrak{X}$ and $\mathbb{P}(\mathfrak{k})$ are the usual ones, so it remains to specify a basis of neighbourhoods of points in $\mathbb{P}(\mathfrak{k})$.

For $0 \le m < \infty$, let $L_m(\ell)$ be the lattice generated by $(\mathfrak{p}^m)^2$ and $\ell \cap \mathfrak{o}^2$. It depends only on the image of ℓ modulo $\mathfrak{p}^m$. For example, $L_0(\ell) = \mathfrak{o}^2$, and if ℓ is the line through $(0,1)$ then $L_m(\ell)$ is the node ν_m. As $m \to \infty$ the lattice $L_m(\ell)$ has as limit the line ℓ.

5.2. Lemma. *Every lattice in $\mathfrak{k}^2$ is equivalent to some $L_m(\ell)$.*

This gives a convenient way to label points of $\mathfrak{X}$. Of course ℓ is uniquely determined only modulo $\mathfrak{p}^m$.

Proof. Suppose L to be a lattice in $\mathfrak{k}^2$. One may assume it to be contained in $\mathfrak{o}^2$. Scaling, one may assume that $\mathfrak{o}^2/L$ has no torsion. Let m be least such that $\varpi^m\mathfrak{o}^2 \subseteq L$. Then the image of L in $(\mathfrak{o}/\mathfrak{p}^m)^2$ is isomorphic to the line generated by some primitive (x, y) in $\mathfrak{o}^2$ (which is unique only modulo $\mathfrak{p}^m$). Let ℓ be the line spanned by (x, y).

By convention, let $L_\infty(\ell) = \ell$. For each $m < \infty$ let $\mathcal{U}_m(\ell)$ be the set of all $L_n(\lambda)$ for $n \ge m$ contained in $L_m(\ell)$. The line ℓ itself lies in this set, as does the lattice $L_m(\ell)$. Let $U_m(\ell)$ be the set of corresponding nodes of $\overline{\mathfrak{X}}$.

I leave it as an exercise to verify:

5.3. Proposition. *The topology defined so as to have the $U_m(\ell)$ as neighbourhoods of ℓ makes the union of $\mathfrak{X}$ and $\mathbb{P}(\mathfrak{k})$ into a compact topological space. The action of $\mathrm{GL}_2(\mathfrak{k})$ on the tree extends to one on this compactification, compatible with that on $\mathbb{P}^1(\mathfrak{k})$.*

As a consequence of Lemma 5.2:

5.4. Proposition. *Every branch with ν_0 as terminus is that associated to some sequence of lattices*

$$L_0(\ell) \subset L_1(\ell) \subset \cdots \subset L_m(\ell) \subset \cdots$$

for some unique line ℓ in $\mathfrak{k}^2$.

5.5. Proposition. *The nodes associated to the $L_m(\ell)$ as ℓ ranges over $\mathbb{P}(\mathfrak{k})$ are the nodes on the circle of radius m with center ν_0.*

In effect, this labelling introduces a mind of polar coordinate system.

Finding a basis of $L_m(\ell)$ depends somewhat on ℓ. This is related to the Bruhat decomposition of $\mathbb{P}(\mathbb{F})$.

5.6. Proposition. *Suppose ℓ to be the span of the primitive vector (x, y). Then:*

(a) if y is a unit, then $(\varpi^m u_0, v_0 + (x/y)u_0)$ is a basis of $L_m(\ell)$;
(b) if x is a unit, then $(u_0 + (y/x)v_0, \varpi^m v_0)$ is a basis of $L_m(\ell)$.

6. Neighbourhood structure and the action of K

The action of $K = \mathrm{SL}_2(\mathfrak{o})$ on $\mathfrak{X}$ is fairly simple.

6.1. Proposition. *Elemernts of the group $\mathrm{SL}_2(\mathfrak{o})$ acts transitively on the nodes of $\mathfrak{X}$ at fixed distance from ν_0.*

Proof. By Proposition 5.5, the points at distance m are the points $L_m(\ell)$ for ℓ in $\mathbb{P}(\mathfrak{k})$. But $\mathbb{P}(\mathfrak{k})$ is the same as $\mathbb{P}(\mathfrak{o})$, so it suffices to see that $\mathrm{SL}_2(\mathfrak{o})$ acts transitively on the primitive pairs (x, y) in $\mathfrak{o}^2$. But the matrix in the proof of Lemma 2.2 can be chosen to have determinant 1.

In other words, the orbits of $\mathrm{SL}_2(\mathfrak{o})$ are the circles around ν_0.

6.2. Proposition. *Elements of the congruence subgroup $\mathrm{SL}_2(\mathfrak{p}^m)$ fixes all points at distance $\le m$ from ν_0. Two nodes at greater distance lie in the same orbit if and only if they are exterior to the same node at distance m from ν_0.*

Proof. Suppose ν a node at distance $n > m$ from ν_0, and that y is the point at distance m from ν_0 on the path to ν_0. By rotation, it may be assumed that $y = \nu_m$. The claim follows from any one of several results proved earlier.

The group $K = SL_2(\mathfrak{o})$ fixes ν_0, representing the lattice $\mathfrak{o}^2$, while its twin $K_* = \alpha^{-1}K\alpha$ fixes its neighbour $\alpha^{-1}(\nu_0) = \nu_{-1}$. Since every compact subgroup fixes some lattice, these two subgroups of $\mathrm{SL}_2(\mathfrak{k})$ are maximal compact. They are not conjugate in SL_2, but in GL_2 conjugation by

$$\begin{bmatrix} \circ & 1 \\ \varpi & \circ \end{bmatrix}$$

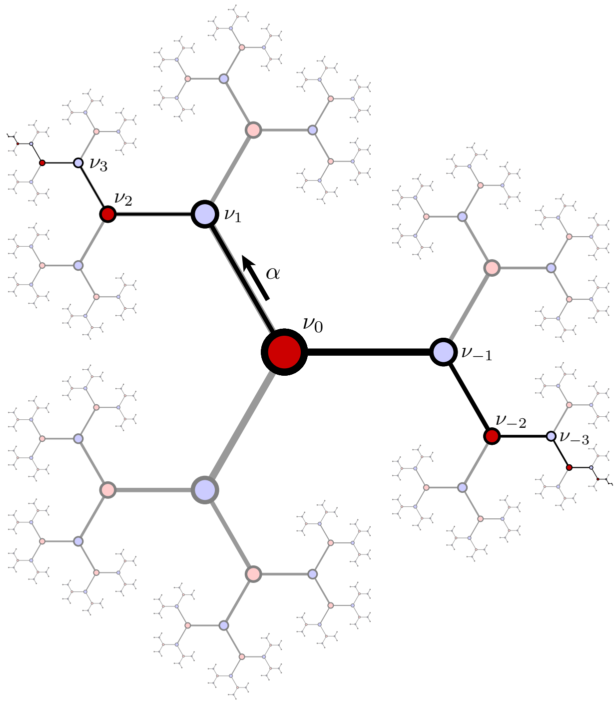
swaps them.

7. Apartments and the action of A

An **apartment** is a doubly infinite geodesic path in $\mathfrak{X}$. It is the union of two branches from one node with no common edge. One apartment is

$$\mathcal{A} = \mathcal{B}_0 \cup \mathcal{B}_\infty = \{\nu_m \,|\, m \in \mathbb{Z}\}\,.$$

Here is a graphical rendering of it:



It is a matter of convention which infinite geodesic I choose to be standard since—as we shall now see—all are equivalent. The choice I have made is convenient for visualization.

Elements of $\mathrm{GL}_2(\mathfrak{k})$ take apartments to apartments.

7.1. Proposition. *The group* $\mathrm{SL}_2(\mathfrak{k})$ *acts transitively on apartments.*

Recall that we are now assuming $\mathfrak{k}$ to be complete.

Proof. It suffices to prove this when one of the apartments is $\mathcal{A}$. Suppose given some other apartment χ, say with two branches χ_0 and χ_∞ running out in opposite directions from the same node ν, which we may assume to have even parity. Since $\mathrm{GL}_2(\mathfrak{k})$ acts transitively on branches, we may transform χ_∞ to the branch $\mathcal{B}_\infty$. Because ν and ν_0 both have even parity, we may assume g to be in $\mathrm{SL}_2(\mathfrak{k})$.

In effect, we may now assume that $\chi_\infty = \mathcal{B}_\infty$. By Lemma 4.3 we may now find a matrix

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with x in $\mathfrak{o}$ that transforms the other branch χ_0 of X into the other branch $\mathcal{B}_0$ of $\mathcal{A}$. But these matrices fix all the nodes on $\mathcal{B}_\infty$, so X is taken to $\mathcal{A}$. ▯

The apartment $\mathcal{A}$ has 0 and ∞ as its ends.

7.2. Corollary. *An apartment is the geodesic path in $\mathfrak{X}$ between its ends ℓ_0, ℓ_∞ in $\mathbb{P}(\mathfrak{k})$. It is taken into itself by the group of matrices having these ends as eigenlines. If x_i is a generator of ℓ_i, then its nodes are the*

$$\langle\!\langle \varpi^m x_0, x_\infty \rangle\!\rangle = \langle\!\langle x_0, \varpi^{-m} x_\infty \rangle\!\rangle \,.$$

Elements of A act as translations on $\mathcal{A}$. The compact subgroup $A(\mathfrak{o})$ acts trivially on it, so the action factors through $A/A(\mathfrak{o})$. The matrix

$$\alpha = \begin{bmatrix} \varpi & \circ \\ \circ & 1 \end{bmatrix}$$

translates ν_m to ν_{m+1}. Since

$$\begin{bmatrix} \varpi^m & \circ \\ \circ & \varpi^{-m} \end{bmatrix} \equiv \begin{bmatrix} 1 & \circ \\ \circ & \varpi^{-2m} \end{bmatrix} \text{ modulo scalar matrices}\,,$$

the subgroup $A \cap \mathrm{SL}_2(\mathfrak{k})$ shifts by an even number of nodes.

The element

$$w = \begin{bmatrix} \circ & -1 \\ 1 & \circ \end{bmatrix}$$

also takes $\mathcal{A}$ to itself, reflecting ν_m to ν_{-m}. The group generated by $A \cap \mathrm{SL}_2$ and w is the normalizer in SL_2 of A. Its quotient by $A(\mathfrak{o})$ is the affine Weyl group W_{aff} of of the root system of SL_2. It is a Coxeter group with elementary reflections w and

$$\begin{bmatrix} \circ & 1/\varpi \\ \varpi & \circ \end{bmatrix} .$$

It contains all reflections in the nodes ν_m of even parity. The segment ν_0 — ν_1 is a strict fundamental domain for the action of W_{aff} on $\mathcal{A}$.

It is not hard to see that the group generated by A and w is precisely the stabilizer of $\mathcal{A}$.

This observation and Proposition 7.1 also imply:

7.3. Proposition. *Given two apartments and an oriented edge in each, there exists g in $\mathrm{GL}_2(\mathfrak{k})$ inducing an isometry of one with the other mapping one oriented edge to the other.*

ORBITS. How does A acts on $\mathfrak{X}$?

We can best describe the action of A in terms of a fibration of $\mathfrak{X}$ with base $\mathcal{A}$. The fibration itself is very simple—it maps a node to the point of $\mathcal{A}$ closest to it. This fibration is A-equivariant, and each fibre is taken to itself by $A(\mathfrak{o})$. So in order to understand how A acts on $\mathfrak{X}$, the main point is to understand how $A(\mathfrak{o})$ acts on a fibre.

For this purpose, it is useful to introduce a family of apartments suggeste by the Iwasawa factorization Corollary 3.3. Recall that every g in GL_2 can be expressed as

$$g = \begin{bmatrix} \varpi^m & x \\ \circ & \varpi^n \end{bmatrix} k = \begin{bmatrix} 1 & x \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \varpi^{m-n} & \circ \\ \circ & 1 \end{bmatrix} \begin{bmatrix} \varpi^n & \circ \\ \circ & \varpi^n \end{bmatrix} k$$

with k in K. Thus

$$g(\nu_0) = \langle\!\langle \varpi^m u_0, v_0 + x u_0 \rangle\!\rangle \,.$$

As $m \to \infty$ this has as limit the point x in $\mathbb{P}(\mathfrak{k})$, and as $m \to -\infty$ it has as limit ∞. As m varies over all of $\mathbb{Z}$ it ranges over all the nodes $L^0_m(x)$ of the apartment $L^0(x)$ between these two boundary points. Coming from x it first strikes $\mathcal{A}$ when $m = \mathrm{ord}(x)$, and then remains in $\mathcal{A}$ for lesser values of m. The fibre over a point ν_k is therefore the union of all the nodes $\langle\!\langle \varpi^m u_0, v_0 + x u_0 \rangle\!\rangle$ with $m \ge k$, $\mathrm{ord}(x) = k$. In particular, the fibre over ν_0 is the union of these points for $m \ge 0$ and units x.

Every other node in $\mathfrak{X}$ has a unique representation as

$$\alpha^k [\![\varpi^m u_0, v_0 + x u_0]\!]$$

in which k is any integer, $m \ge 0$ and x is a unit, unique modulo $1 + \mathfrak{p}^m$. Elements

$$\begin{bmatrix} a & \circ \\ \circ & d \end{bmatrix}$$

of $A(\mathfrak{o})$ act on this in a simple fashion: $x \mapsto ax/d$.

7.4. Proposition. *Two nodes in the tree lie in the same A-orbit if and only if they lie at the same distance from $\mathcal{A}$.*

7.5. Proposition. *Two nodes in the tree lie in the same $A(\mathfrak{p}^n)$-orbit if and only if they lie on the same lateral branch attached to $\mathcal{A}$ at the same distance from $\mathcal{A}$ and the subset of points in common on the paths to $\mathcal{A}$ have length at least n.*

This will turn out to be a basic tool in computing orbital integrals.

7.6. Proposition. *Suppose a to lie in $A(\mathfrak{p}^n)$ but not in $A(\mathfrak{p}^{n+1})$. Then a fixes the node ν if and only if it lies at distance $\le n$ from $\mathcal{A}$.*

The analogous results for $\mathrm{SL}_2(\mathfrak{k})$ can be deduced from this and the observation that

$$\begin{bmatrix} t & \circ \\ \circ & 1/t \end{bmatrix} \sim \begin{bmatrix} t^2 & \circ \\ \circ & 1 \end{bmatrix} .$$

The apartments $L^0(x)$ define a certain retraction ρ of $\mathfrak{X}$ onto $\mathcal{A}$. It takes $L^0_m(x)$ to ν_m. In particular it maps any node on $\mathcal{A}$ to itself.

7.7. Lemma. *The map ρ is N-invariant:*

$$\rho(n(\nu)) = \rho(\nu) \,.$$

8. The action of N

As $m \to \infty$ the lattice $[[1, \varpi^m]]$ passes off to the line through $(1, 0)$ in $\mathbb{P}^1(\mathfrak{k})$. In other words, the group N of all upper triangular unipotent matrices fixes the end of the branch $\{\nu_m \mid m \le 0\}$, which amounts to ∞ in $\mathbb{P}^1(\mathfrak{k})$. There is a finite approximation of this phenomenon. Recall that $N(\mathfrak{p}^m)$ is the subgroup of

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

with $x \in \mathfrak{p}^m$. The following is elementary, but useful to refer to.

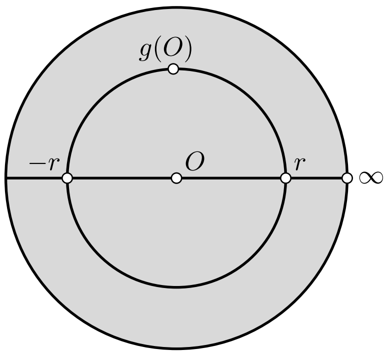
8.1. Proposition. *(a) Elements in $N(\mathfrak{p}^{-m})$ fix all nodes ν_{-k} with $k \ge m$. (b) The $N(\mathfrak{p}^{-m})$-orbit of ν_{-k} for $k < m$ is the set of points x at distance $m - k$ from ν_{-m} other than those on a path starting back to ν_{-m-1}.*

As $m \to \infty$ the group $N(\mathfrak{p}^{-m})$ expands, consistently with what happens at the end point. Claim (a) is trivial. For (b) look at Lemma 4.3.

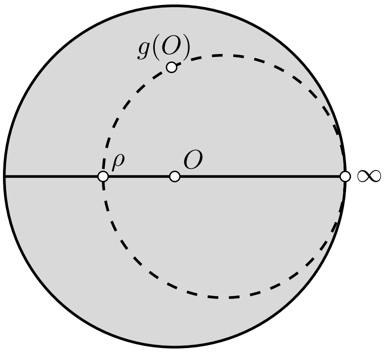
There is an important relation between the Cartan and Iwasawa factorizations. I recall first what happens for $G = \mathrm{SL}_2(\mathbb{R})$. Let K be $\mathrm{SO}(2)$, N be the N is the group of unipotent upper triangular matrices, A be the group of diagonal matrices, and $P = AN$.

The **Cartan factorization** asserts that $G = K\boldsymbol{A}^{++}K$. Geometrically things are simple. We first represent G by Möbius transformations of the unit disk, conjugating the more familiar action on the upper half plane by the Cayley transform. If $g = k_1 a k_2$ then it is also $k_1 a^{-1} k_2$. Choose a so that $r = a(O)$ lies in the interval $(0, 1)$. Then $g(P) = k_1 a(O)$ will lie at angle -2θ on the circle of radius r around O if

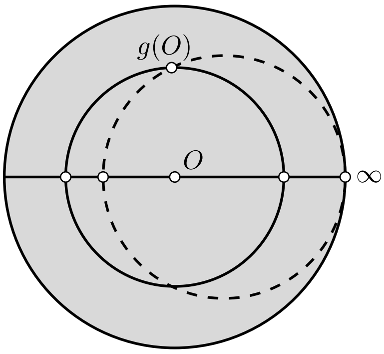
$$k_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$



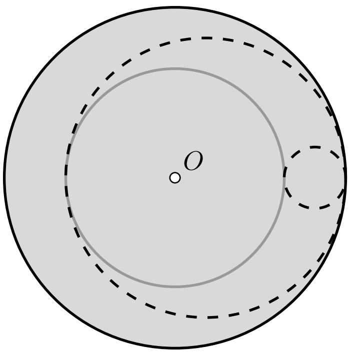
The **Iwasawa factorization** asserts that $G = NAK$. There is a simple geometric description here, too. If $g = nak$ then $g(O) = na(O)$, which is on the N-orbit of r. The N-orbits are the circles inside the unit disk and tangent to 1. So we find the circle of this sort which passes through $g(O)$, and then find where it intersects the real line.



How do these different factorizations compare? The answer is that if $g(O)$ lies on the circle of radius r around O and on the N-orbit through ρ then $-r \le \rho \le r$.



Verifying this is very easy, given the following picture. Keep in mind that orbits can't cross, so a simple continuity argument will do.



Here is the generalization of this for $\mathrm{SL}_2(\mathfrak{k})$:

8.2. Proposition. *Suppose g in $\mathrm{GL}_2(\mathfrak{k})$. Then*

- *(a) if $g = nak$ is its Iwasawa factorization, then it has Cartan factorization $g = k_1 d k_2$ with $a(\nu_0)$ in the convex hull in $\mathcal{A}$ of (a.k.a. line segment between) $d(\nu_0)$ and $d^{-1}(\nu_0)$;*
- *(b) if ν lies in $\mathcal{A}$ then the intersection of its K-orbit and N-orbit is just ν itself.*

The proof of the Proposition applies the retraction ρ constructed in the last section.

Proof. Suppose $g = nak$. Let ρ be the retraction referred to in Lemma 7.7. Then the image of a in $\mathcal{A}$ is $\rho(x)$. The matrix d is determined by the geodesic from ν_0 to x, and the image of this path under ρ has length at least that of ρ. But this is exactly what the Proposition asserts.

I leave claim (b) as an exercise.

A generalization of the result for arbitrary real semi-simple groups has been proved in [Kostant:1973], and this in turn has been generalized in [Atiyah:1982]. A first step towards a generalization of this for $\mathfrak{p}$-adic groups can be found in §4.4 of [Bruhat-Tits:1972] (see also Theorem 2.6.11(3)–(4) of [Macdonald:1971]), and the precise $\mathfrak{p}$-adic analogue of Kostant's result can be found in [Hitzelberger:2010].

A FILTRATION. One feature of the apartment $\mathcal{A}$ that becomes more significant for groups of higher rank is that its structure mirrors that of the unipotent subgroup of upper triangular matrices. This group is filtered by subgroups

$$N(\mathfrak{p}^n) = \left\{ \begin{bmatrix} 1 & \mathfrak{p}^n \\ \circ & 1 \end{bmatrix} \right\} ,$$

and the set of points on $\mathcal{A}$ fixed by this subgroup consists of all those on the branch

$$\nu_n - \nu_{n-1} - \nu_{n-2} - \cdots .$$

In other words, the filtration of N by the groups $N(\mathfrak{p}^n)$ is compatible with that of branches in $\mathcal{A}$ passing off to ∞, in one branch is contained in another if and only if the subgroups of N fixing nodes in the first contains that fixing nodes in the second. This is a crucial feature of all Bruhat-Tits buildings.

The case

$$g = \begin{bmatrix} 1 & \circ \\ x & \varpi^m \end{bmatrix}$$

can be dealt with similarly.

9. Iwahori subgroups

9.1. Proposition. *Suppose*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

The following are equivalent:

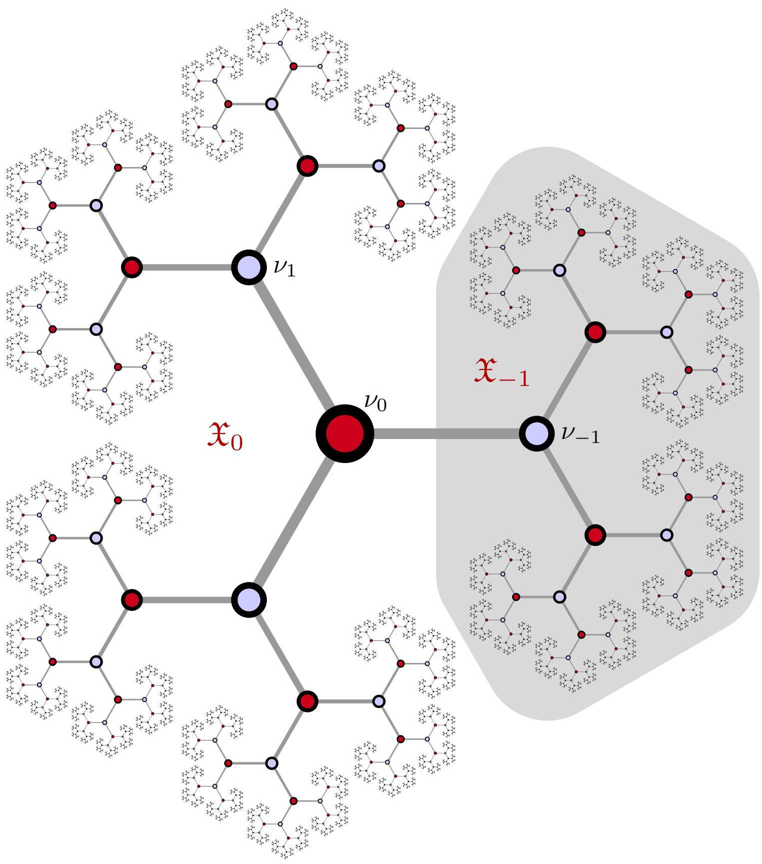
(a) $c \equiv 0 \bmod \mathfrak{p}$;
(b) g *fixes the lattice flag*

$$[\![u_0, \varpi v_0]\!] \subset [\![u_0, v_0]\!] ;$$

(c) g *fixes all points on the edge* $\nu_0 — \nu_{-1}$*;*
(d) g *lies in the intersection* $K \cap \alpha^{-1} K \alpha$*.*

Let I be the group of all such matrices. Its conjugates are called **Iwahori subgroups**. Each acts trivially on exactly one edge of $\mathfrak{X}$. The quotient $\mathrm{PGL}_2(\mathfrak{k})/I$ may therefore be identified with the oriented edges of the building.

Let $\mathfrak{X}_{-1}$ be the set of nodes in $\mathfrak{X}$ lying in the exterior of ν_{-1}, the shaded region in the following figure. Let $\mathfrak{X}_0$ be its complement.



This partition is also a consequence of Proposition 3.8. The group I takes each of these regions into itself.

The union of ν_0 and $\mathfrak{X}_{-1}$ is the subset of nodes in $\mathfrak{X}$ fixed by $N(\mathfrak{o})$.

Because of Proposition 4.2:

9.2. Proposition. *The orbits of I among the nodes of the building are the points at a fixed oriented distance from the edge $\nu_0 — \nu_1$.*

In particular:

9.3. Corollary. *Each I-orbit among the nodes of $\mathfrak{X}$ intersects $\mathcal{A}$ in exactly one node.*

This establishes a bijection between the orbits of I and the nodes of $\mathcal{A}$.

The orbits may be described more geometrically. Those through ν_m with $m \geq 0$ are the endpoints of the paths of length m leading out from ν_0 that do not start with that through ν_1. Those through ν_{-m} with $m > 0$ are the endpoints of paths of length m that do pass through ν_1.

As one consequence of the description of I orbits:

9.4. Proposition. *Given an apartment $\mathcal{A}$ and a chamber C in it, there exists a unique map $\rho = \rho_{\mathcal{A},C}$ from the tree onto $\mathcal{A}$ with these properties:*

- *(a) ρ is the identity on $\mathcal{A}$;*
- *(b) it is an isometry on every apartment containing C.*

If I_C is the Iwahori subgroup fixing C, then $\rho(bx) = \rho(x)$ for all b in I_C, x in the tree.

Proof. The proof is geometric. Choose a point y in the middle of C. If x is an arbitrary point in the building, there exists a unique geodesic from x to y. But there also exists a unique geodesic of the same length in $\mathcal{A}$ that agrees with the first for points inside C. Map x to its endpoint.

It is interesting to figure out how to compute the retraction ρ for the standard apartment $\mathcal{A}_0$ and chamber e_0. That is to say, given g in $\mathrm{PGL}_2(\mathfrak{k})$, the retraction of $\rho(ge_0)$ will be an edge in $\mathcal{A}_0$. Which?

I'll answer this in a strong sense by orienting the edges, with e_0 going from ν_0 to ν_{-1}. Here the answer is given by a result that is important in representation theory. First of all, any oriented edge of $\mathcal{A}_0$ is the transform of an element in the subgroup Ω of matrices of the form

$$\begin{bmatrix} \varpi^m & \circ \\ \circ & \varpi^n \end{bmatrix}, \begin{bmatrix} \circ & \varpi^n \\ \varpi^m & \circ \end{bmatrix}$$

which is unique up to a scalar multiple.

If $g = b_1\omega b_2$ with the b_i in I and ω in Ω, then $\rho g e_0$ is equal to ωe_0. The proof of the following result will tell us how to find ω.

9.5. Proposition. *Every g in $\mathrm{GL}_2(\mathfrak{k})$ may be expressed as a product $b_1\omega b_2$ with the b_i in I, ω in Ω.*

Proof. To go with this claim is an algorithm involving **elementary Iwahori operations** on columns:

- Add to a column d a multiple xc of a previous column c by some x in $\mathfrak{o}$;
- add to a column c a multiple xd of a subsequent column by x in $\mathfrak{p}$;
- multiply a column by a unit in $\mathfrak{o}$;

and also on rows:

- Add to a row c a multiple xd of a subsequent row d with x in $\mathfrak{o}$;
- add to a row d a multiple xc of a previous row with x in $\mathfrak{p}$;
- multiply a row by a unit in $\mathfrak{o}$;

Each of these column (row) operations amounts to right (resp. left) multiplication by what I'll call an Iwahori matrix .

Here are some examples:

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u & xu + v \end{bmatrix}$$

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varpi x & 1 \end{bmatrix} = \begin{bmatrix} u + \varpi x v & v \end{bmatrix}$$

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u + xv \\ v \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \varpi x & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ \varpi x u + v \end{bmatrix} .$$

9.6. Proposition. *Any invertible 2×2 matrix can be reduced by elementary Iwahori row and column operations to a unique matrix in Ω.*

Let m be the given matrix. First of all, it is easy to apply elementary row operations to obtain a matrix of the form

$$\begin{bmatrix} \varpi^k & *\varpi^n \\ \circ & \varpi^\ell \end{bmatrix}, \quad \begin{bmatrix} \circ & \varpi^\ell \\ \varpi^k & *\varpi^n \end{bmatrix} .$$

Let's look at the two cases separately.

(1)

$$g = \begin{bmatrix} \varpi^k & u\varpi^n \\ 0 & \varpi^\ell \end{bmatrix} .$$

If $k \le n$ we can subtract $u\varpi^{n-k}$ times the first column from the second to get

$$\begin{bmatrix} \varpi^k & 0 \\ 0 & \varpi^\ell \end{bmatrix} .$$

If $\ell \le n$ we can subtract $u\varpi^{n-\ell}$ times the second row from the first to get the same matrix.

So now we may assume $k > n$ and $\ell > n$. Subtract from the second row $\varpi^{\ell-n}/u$ times the first. This gives

$$\begin{bmatrix} \varpi^k & u\varpi^n \\ -\varpi^{k+\ell-n}/u & 0 \end{bmatrix} .$$

Divide the second column by u, multiply the second row by $-u$:

$$\begin{bmatrix} \varpi^k & \varpi^n \\ \varpi^{k+\ell-n} & 0 \end{bmatrix} .$$

Subtract ϖ^{k-n} times the second column from the first to get

$$\begin{bmatrix} 0 & \varpi^n \\ \varpi^{k+\ell-n} & 0 \end{bmatrix} ,$$

which is Iwahori reduced.

(2)

$$g = \begin{bmatrix} 0 & \varpi^\ell \\ \varpi^k & u\varpi^n \end{bmatrix} .$$

where u is a unit.

If $n > \ell$ or $n \geq k$ this can be reduced to

$$\begin{bmatrix} 0 & \varpi^\ell \\ \varpi^k & 0 \end{bmatrix} .$$

So now we assume $n \leq \ell$, $n < k$. Subtract ϖ^{k-n}/u times the second column from the first:

$$\begin{bmatrix} -\varpi^{k+\ell-n}/u & \varpi^\ell \\ 0 & u\varpi^n \end{bmatrix} .$$

Multiply and divide by u:

$$\begin{bmatrix} \varpi^{k+\ell-n} & \varpi^\ell \\ 0 & \varpi^n \end{bmatrix} .$$

Subtract $\varpi^{\ell-n}$ times the second row from the first:

$$\begin{bmatrix} \varpi^{k+\ell-n} & 0 \\ 0 & \varpi^n \end{bmatrix} .$$

Remark. For groups of higher rank, the Iwahori subgroups are the stabilizers of the simplices in the Bruhat-Tits building of maximal dimension. The analogue of Proposition 9.5 is important in representation theory.

10. *Appendix.* A fixed point theorem

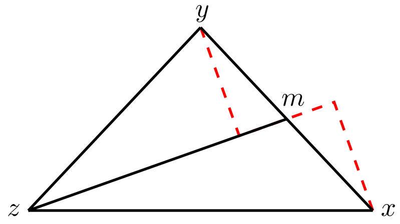
For any two points x, y in the tree, let $m_{x,y}$ be the midpoint of the geodesic between them. The following asserts that in some sense the tree has non-positive curvature.

10.1. Proposition. (Bruhat-Tits inequality) *Given points x, y, and z on the tree, let $m = m_{x,y}$. Then*

$$|z{:}m|^2 + |m{:}x|^2 \le \frac{|z{:}x|^2 + |z{:}y|^2}{2} \,.$$

Keep in mind that $|m{:}x| = |m{:}y|$.

Proof. This is an equality on an apartment, according to a theorem of Pappus. It is an easy vector calculation, or can be proved by applying Pythagoras' Theorem a few times.



In general, fix an apartment $\mathfrak{A}$ containing x and y and let E be an edge containing m. If ρ is the retraction determined by $\mathfrak{A}$ and E, then

$$|z{:}x|^2 + |z{:}y|^2 \ge |\rho(z){:}x|^2 + |\rho(z){:}y|^2 = 2|\rho(z){:}m|^2 + 2|m{:}x|^2 = 2|z{:}m|^2 + 2|m{:}x|^2 \,.$$

Any metric space satisfying this condition is called **semi-hyperbolic**. All Bruhat-Tits buildings and all non-compact real symmetric spaces fall in this category. Any two points on a semi-hyperbolic space have a unique midpoint between them. The sphere, for example, is not semi-hyperbolic.

If X is any bounded set in the tree and c a point in the tree, there exists $R \ge 0$ such that $|c{:}x| < R$ for all x in X. Define $R_c(X)$ to be the least upper bound of all such R, and define the radius R_X of X to be the least upper bound of all $R_c(X)$ as c varies. A circumcentre for X is a point c with the property that $|c{:}x| \le R_X$ covers X. The following is an observation due to Serre.

10.2. Corollary. *Every bounded subset of the tree has a unique circumcentre.*

Proof. Choose a sequence c_i such that $R_{c_i}(X) \to R(X)$. The semi-hyperbolic inequality implies that it is a Cauchy sequence.

The case we shall be interested in is that in which X is a finite set. Is there a simple algorithm to find its circumcentre?

We have now a new proof of a result we have seen before. In contrast to the earlier proof, this one can be expanded into one for all buildings.

10.3. Corollary. *Any compact subgroup of $\mathrm{SL}_2(\mathfrak{k})$ fixes some point on the tree.*

Proof. Because it fixes the circumcentre of any orbit.

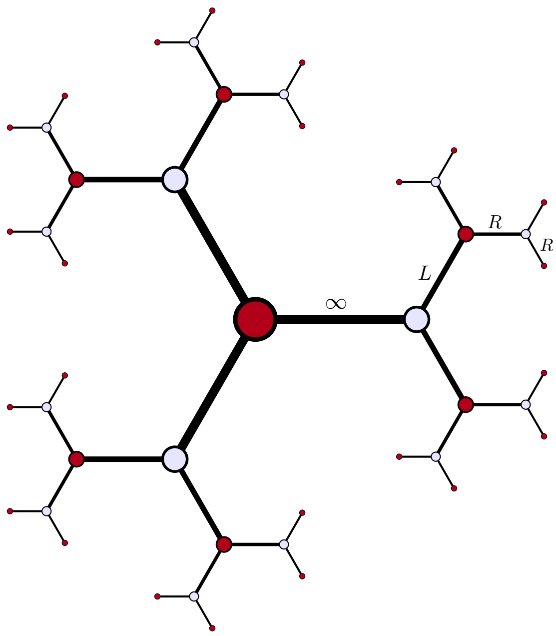
Hence the subgroups fixing nodes of the tree are maximal compact subgroups of $\mathrm{SL}_2(\mathfrak{k})$, and there are two conjugacy classes of them. For $\mathrm{PGL}_2(\mathfrak{k})$ there is just one.

11. *Appendix.* Navigating in the tree

I'll discuss here how to draw the tree for $G = \mathrm{SL}_2(\mathbb{Q}_2)$. This can be done at several levels of sophistication.

First I'll describe how to draw the basic tree. This is the tree simply as a geometric object, a collection of branches, and no association with an automorphism group. There are a number of parameters that determine it—the dimensions of nodes and edges, how these should shrink with depth, how edges are arrayed around a node, and colour choice. The drawing is then done by recursion, either explicitly or implicitly, with a stack, out to some given depth. Each node is assigned an angle in, as well as location. A node draws itself, and if the specified depth has not been exceeded it then draws edges out to neighbours, and finally draws those neighbours by recursion. I'll leave details as an exercise.

Still on the purely geometric level of drawing is a procedure for drawing nodes along a path like ∞LRR as indicated in this figure:



So it is relatively simple to draw the tree as a geometric object. But for really useful (i.e. 'intelligent') drawings we want to translate back and forth between nodes in the drawing and lattices, or between nodes and elements of G. That is to say, suppose $\mathfrak{o} = \mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at (2). We use the action of $\mathrm{GL}_2(\mathbb{Q})$ on the tree, rather than that of the 2-adic field, because it is computationally feasible. The nodes in the tree are the same for localizations as for completions, and pretty much the only difference between the two groups is that the group over $\mathbb{Q}$ is smaller and does not act transitively on apartments.

In other words, we want to associate to each node in the geometric tree a 2×2 invertible matrix in $\mathrm{PGL}_2(k)$, and vice-versa. This means building a bijection between certain g and paths like ∞LRR as explained above. This is easily done by applying . This tells us that the nodes in $\mathfrak{X}_0$ are of the form $n\alpha^m\nu_0$ with n in $N(\mathfrak{o})$ and $m \geq 0$, while those in $\mathfrak{X}_{-1}$ are of the form $\overline{n}\alpha^{-m}\nu_0$ with $\overline{n}$ in $\overline{N}(\mathfrak{p})$ and $m > 0$. The $N(\mathfrak{o})$-orbit of nu_m $(m \geq 0)$ is in bijection with $N(\mathfrak{o})/N(\mathfrak{p}^m)$, and the $\overline{N}(\mathfrak{p})$-orbit of ν_m $(m < 0)$ is in bijection with $\overline{N}(\mathfrak{p})/\overline{N}(\mathfrak{p}^m)$.

Now take $q = 2$. The nodes of the tree are parametrized by sequences of L and R, either from the root node if $n \leq 0$ or from the infinite node if $n > 0$. So we must define the map from the section matrices parametrized by x modulo $\mathfrak{p}^n$ to such a sequence, and vice-versa.

Suppose $n > 0$. We are given x as an even integer $2y$ modulo 2^n. We find the bits of y and read from low order $i = 0$ up to order $i = n - 1$, translating bit i:

$i = 0, 2, 4, \ldots$	odd $\mapsto L$,	even $\mapsto R$,
$i = 1, 3, 5, \ldots$	even $\mapsto L$,	odd $\mapsto R$

Now suppose $n \le 0$. We are given x as an integer modulo $2^{|n|}$. We find the bits of x and read from low order $i = 0$ up to order $i = |n| - 1$, translating bit i:

$i = 0, 2, 4, \ldots$	odd $\mapsto L$,	even $\mapsto R$
$i = 1, 3, 5, \ldots$	even $\mapsto L$,	odd $\mapsto R$

In short, the rules are the same! They can be summarized in a table:

bit index parity	bit parity	L or R
0	0	R
0	1	L
1	0	L
1	1	R

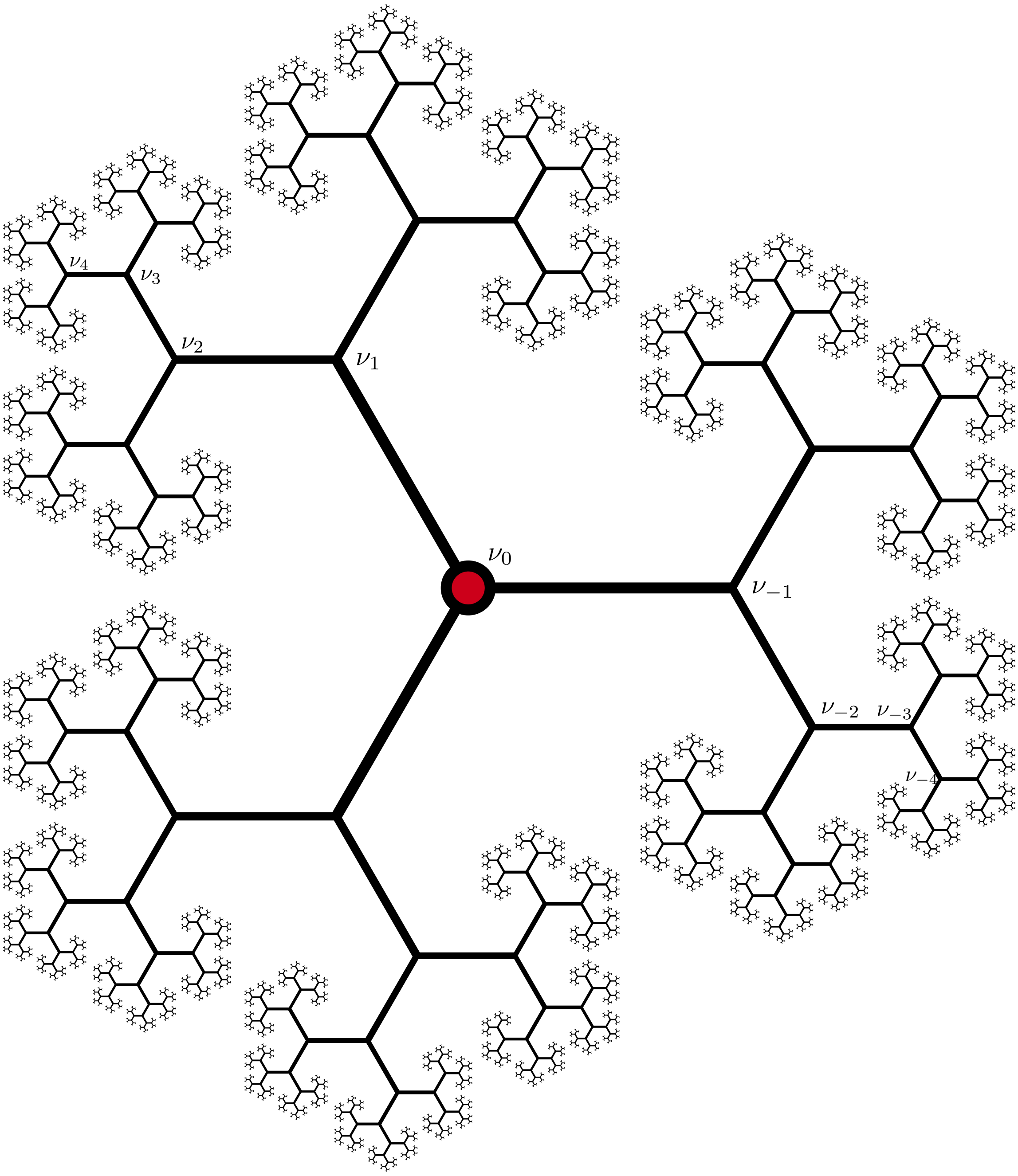
But now you can see that they can be formulated most succinctly as addition modulo 2, with $R = 0$, $L = 1$.

One final remark—it might seem at first that the map between LR paths and nodes is somewhat arbitrary. But in fact some labelings are better than others, in the sense that the geometry of the action of G looks more or less comprehensible. The one I have chosen here seems to be best. One reason for this is that the geometry of the orbits the matrices

$$\begin{bmatrix} t & \circ \\ \circ & 1/t \end{bmatrix}$$

is simple. It does have one strange feature, however: the action of A on the apartment $\mathcal{A}$ swaps sides as it translates.

12. *Appendix.* **Centrefold**



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