Essays on Coxeter groups

Coxeter elements in finite Coxeter groups

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A finite Coxeter group possesses a distinguished conjugacy class of **Coxeter elements**. The literature about these is very large, but it seems to me that there is still room for a better motivated account than what exists. The standard references on this material are [Bourbaki:1968] and [Humphreys:1990], but my treatment follows [Steinberg:1959] and [Steinberg:1985], from which the clever parts of my exposition are taken. One thing that is somewhat different from the standard treatments is the precise link between Coxeter elements and the Coxeter-Cartan matrix.

This essay si incomplete. In particular, it does not discuss the formula for the order of a Coxeter element in terms of the number of roots, or the relations between eigenvalues of Coxeter elements and polynomial invariants of the Group W.

Contents

1. Definition

Suppose (W, S) to be an irreducible finite Coxeter system with $|S| = n$. The group W is generated by elements s in S, with defining relations $(s_i s_j)^{m_{i,j}} = 1$. The s in S are of order two, so each diagonal entry $m_{i,i}$ is equal to 1. The matrix $(m_{i,j})$ is called the **Coxeter matrix**.

The corresponding **Coxeter graph** has as its nodes the elements of S, and there is an edge between two elements if and only if they do not commute. Because (W, S) is irreducible, this graph is connected.

For example, the Weyl group of a finite root system is a Coxeter group. The generators in S are the reflections associated to a set of simple roots α_i. If $\mathfrak{C}$ is the corresponding Cartan matrix $(\langle \alpha_i, \alpha_j^\vee \rangle)$, the Coxeter matrix is determined by the rule

$$m_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ (when } \langle \alpha_i, \alpha_j^\vee \rangle = 2) \\ 2 & \quad \langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = 0 \\ 3 & \quad \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = 1 \\ 4 & \quad \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = 2 \\ 6 & \quad \langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle = 3. \end{cases}$$

In general, the group W is realized as a group of orthogonal transformations on $\mathbb{R}^n$, generated by reflections s in S. Fix a W-invariant positive definite inner product $u \bullet v$. Let C be one of the two connected components of the complement of the reflection hyperplanes fenced in by the hyperplanes fixed by the s_i. For each s in S let a_s be the unit vector that is positive on C and taken into $-a_s$ by s. The corresponding reflection is then

$$s\colon\ v \longmapsto v - 2(a_s \bullet v)a_s\,,$$

and

$$C = \{v \mid a_s \bullet v > 0 \text{ for all } s \in S\}\,.$$

The reflections s_i and s_j are linked in the graph if and only if $a_i \bullet a_j = -\cos(\pi/m_{i,j}) \neq 0$. Let $\mathfrak{A}$ be the symmetric positive definite matrix $(a_{i,j}) = (a_i \bullet a_j)$. Thus $a_{i,i} = 1$, $a_{i,j} \le 0$ for $i \neq j$, and $a_{i,j} \le -1/2$ if it is < 0. The matrix $2\mathfrak{A}$ is what I call elsewhere the Cartan matrix of the given realization of W as a group generated by reflections. The matrix $\mathfrak{A}$ is the matrix of the given W-invariant quadratic form with respect to the basis a_i. That is to say, if $u = \sum u_i a_i$ then

(1.1)
$$\|u\|^2 = {}^t u\,\mathfrak{A}\,u\,.$$

When W is the Weyl group of a simple root system, there is a simple relationship between the two Cartan matrices. First of all one must choose an invariant quadratic form on the root system. I'll offer are two ways to do this.

(1) First fix the length of one root, say setting $\|\alpha_1\| = 1$. If α_i and α_j are neighbours in the Dynkin diagram, orthogonality of reflections imposes the condition

$$\langle \alpha_i, \alpha_j^\vee \rangle = 2\left(\frac{\alpha_i \bullet \alpha_j}{\alpha_j \bullet \alpha_j}\right)\,.$$

Given $\|\alpha_i\|^2$, this allows us to deduce for each neighbour

$$\begin{aligned} \alpha_i \bullet \alpha_j &= (1/2)\langle \alpha_j, \alpha_i^\vee \rangle \|\alpha_i\|^2 \\ \|\alpha_j\|^2 &= 2\left(\frac{\alpha_i \bullet \alpha_j}{\langle \alpha_i, \alpha_j^\vee \rangle}\right)\,. \end{aligned}$$

In effect

$$\mathfrak{C} D = 2\mathfrak{A}$$

if D is the diagonal matrix with entries $\|\alpha_i\|^2$.

(2) We have

$$u \bullet v = \sum\nolimits_{\lambda>0} \langle u, \lambda^\vee \rangle \langle v, \lambda^\vee \rangle\,.$$

In any event, the vector a_i is the unit vector $\alpha_i/\|\alpha_i\|$, so

$$\sum c_i \alpha_i = \sum c_i \cdot \|\alpha_i\| \cdot a_i\,.$$

1.2. Lemma. *The Coxeter graph of an irreducible finite Coxeter group is a tree.*

Proof. Suppose we have a simple circuit μ_i in Δ, with $\mu_i \bullet \mu_{i+1} < 0$, $\mu_n \bullet \mu_1 < 0$. Let

$$\lambda = \sum \mu_i$$

Then

$$\lambda \bullet \lambda = n + 2\sum_{i<j} \mu_i \bullet \mu_j\,.$$

But

$$\mu_i \bullet \mu_{i+1} \le -1/2, \quad \mu_1 \bullet \mu_n \le -1/2$$

so the second term is $\le -n$, and $\lambda \bullet \lambda \le 0$. ▯

1.3. Lemma. *If T is any finite tree, its nodes may be numbered in such a way that nodes 1 to m form a connected sub-tree of which node m is a leaf.*

Proof. The proof will lay out an algorithm in which every node is assigned an index conforming with the assertion. The initial data will list for every node all of its neighbours.

In fact, for future use, *the algorithm will do more—it will assign to each node a height, which measures its distance from the initial node chosen. This will assign a parity to each node, namely the parity of its height, with the property that the neighbours of a node have different parity from it. The algorithm will also assign to each node of index m the node of index less than m to which it is attached.*

Start with an arbitrary node as root. Every other node will be assigned height equal to its distance from this root node, and at the end the parity of a node will be the parity of its height. Assign the root node index 0 and height 0, and put it on a stack. being on the stack means that a node has been assigned a height, but that its neighbours have not necessarily been examined. While the stack is not empty, pop a node x from it. Scan through its neighbours for those neighbours y to which height and index have not been assigned. If x has height η assign y height $\eta+1$, and if m was the last index assigned, assign y index $m+1$. Assign x as the ancestor of y, and put y on the stack.

Suppose w to be any product $s_{i_1}\ldots s_{i_m}$ of elements of S. There are two elementary operations that transform w into a conjugate. First of all, w is conjugate to $s_{i_m} w s_{i_m} = s_{i_m} s_{i_1} \ldots s_{i_{m-1}}$. This shift from the right end to the left, or the one in the opposite direction, I'll call an **end-shift**. Second, if s_{i_k} and $s_{i_{k+1}}$ commute, we can replace $s_{i_k}s_{i_{k+1}}$ by $s_{i_{k+1}}s_{i_k}$ without changing w. I'll call this a **swap**.

Repeated end-shifts allow us to shift any segment to either end. For example:

$$\begin{aligned} s_1 s_2 s_3 s_4 s_5 &\sim s_2 s_3 s_4 s_5 s_1 \\ &\sim s_3 s_4 s_5 s_1 s_2 \,. \end{aligned}$$

In the following, assume the elements of S to be numbered according to Lemma 1.3. What this means is that, for each m, all s_i with $i < m-1$ commute with s_m.

1.4. Lemma. *If $\gamma = s_{i_1}\ldots s_{i_m}$ $(m \le n)$ with all s_i distinct, then any permutation of this word may be obtained from it by a sequence of end-shifts and swaps.*

Proof. More precisely, I am going to lay out an algorithm that will transform any Coxeter element into $s_1 \ldots s_n$. This will be done, roughly speaking, by induction on m, but in a slightly tricky way. For $n = 1$ or 2 the result is immediate.

What we are going to do is produce in stages. At the k-th stage, we shall apply end-shifts and swaps to obtain a word in which the last k terms make up the product $s_{n-k+1}\ldots s_n$.

Step 1. The first step is simple. By a succession of end-shifts we can move s_n right to the end of the word, so now we have

$$s_{i_1}\ldots s_{i_{n-1}} s_n \text{ to } s_{i_{n-1}} s_{i_1}\ldots s_{i_{n-1}} s_n \,,$$

with all $i_k < n$.

Step 2. We now want to get a new word in which the final pair is $s_{n-1}s_n$. I do this by induction on the distance in the word separating s_{n-1} from s_n. **(a)** If this is 0 then $i_{n-1} = n-1$, there is nothing to do. **(b)** Otherwise, s_{n-1} sits inside the first $n-2$ terms, and we want to shift it right while keeping s_n at the right end. The node s_n is a leaf of the tree of the first n nodes, so there is at most one other node $s = s_i$ with $i < n$ linked to it. **(b)(i)** If $s_{i_{n-1}} \ne s$, we can swap it with s_n and then shift it around to the beginning. We have thus decreased the distance between s_n and s_{n-1} by 1, and s_n does remain at the right end. Apply the induction hypothesis (or, in a program, loop). **(b)(ii)** Otherwise, $s_{i_{n-1}} = s$. But then s_n commutes with all s_{i_k} for all $1 \le k < n-1$, so we may shift s_n around to the left, and then by a succession of swaps get s_n just to the right of s_{n-1}. Finally, apply end-shifts to move the pair $s_{n-1}s_n$ to the right.

Step 3. We keep on going. At stage $k \geq 2$, we have a word of the form

$$s_{i_1} \dots s_{i_{n-k}} \cdot s_{n-k+1} \dots s_n$$

and we want to get to a word of the form

$$s_{i_1} \dots s_{i_{n-k-1}} \cdot s_{n-k} s_{n-k+1} \dots s_n \, .$$

The basic operation is to transform the word

$$s_{i_1} \dots s_{i_{n-k}} \cdot s_{n-k+1} \dots s_n$$

by effecting an **interior end-shift**, changing this to

$$s_{i_{n-k}} s_{i_1} \dots s_{i_{n-k-1}} \cdot s_{n-k+1} \dots s_n \, ,$$

and thus decreasing the distance between s_{n-k} and s_{n-k+1}.

We do this by recursion on k, the length of the terminal word. We have seen that we can do this for $k = 0$ or 1. Suppose $k > 1$. The node s_{n-k+1} is linked to exactly one of the earlier nodes. **(a)** If this node is not $s_{i_{n-k}}$, we can swap it with s_{n-k+1}, getting the new word

$$s_{i_1} \dots s_{n-k+1} s_{i_{n-k}} \cdot s_{n-k+2} \dots s_n$$

But now we make a recursive call, with a terminal word of length $k-1$. **(b)** Otherwise, suppose this node to which s_{n-k+1} is linked is in fact its neighbour. We again make a recursive call to get

$$s_{n-k+1} s_{i_1} \dots s_{i_{n-k}} \cdot s_{n-k+2} \dots s_n \, ,$$

followed by a succession of swaps to get

$$s_{i_1} \dots s_{n-k+1} s_{i_{n-k}} \cdot s_{n-k+2} \dots s_n \, ,$$

which is in turn changed by a recursive call to get

$$s_{i_{n-k}} s_{i_1} \dots s_{i_{n-k-1}} \cdot s_{n-k+1} s_{n-k+2} \dots s_n \, .$$

A **Coxeter element** of W is any product of all n elementary reflections in S. As an immediate consequence:

1.5. Theorem. *All Coxeter elements are conjugate in W.*

If s and t have indices of the same parity, they will commute. Hence another consequence of the algorithm outlined in the proof of Lemma 1.3:

1.6. Proposition. *There exists a Coxeter element of the form*

$$w = xy$$

where

$$x = s_{k_1} \dots s_{k_m}, \quad y = s_{k_{m+1}} \dots s_{k_n}$$

and all the s_k in each product commute with each other.

Thus x and y are both involutions. I shall call such an element a **distinguished Coxeter element**.

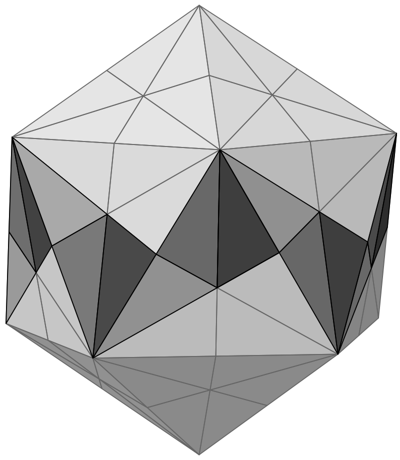
Remark. The results in this section remain valid for any Coxeter group whose graph is a tree.

2. The icosahedron

An example will be instructive. Suppose W to be the symmetry group of the icosahedron, with $|W| = 120$. The fundamental domain of W acting on the icosahedron is a triangle on one of the faces, and W is generated by the three reflections s_i in its walls, with relations (say)

$$s_1 s_2 = s_2 s_1, \quad (s_2 s_3)^3 = 1, \quad (s_1 s_3)^5 = 1 \,.$$

Let γ be the Coxeter element $s_1 s_2 s_3$. The orbit of triangles with respect to the group generated by γ is an equatorial belt around the icosahedron, as shown below:



In this figure, triangles in the same orbit are coloured the same. Thus γ acts as a rotation by $2\pi/10$ in the equatorial plane, and swaps poles. The eigenvalues of γ are therefore $e^{2\pi i/10}$, -1, and $e^{-2\pi i/10}$.

3. Eigenvalues and eigenvectors

If $W = W_S$ is a Coxeter group, then subsets $T \subseteq S$ gives rise to Coxeter subgroups W_T of W, generated by reflections in T, which all fix non-trivial faces of a fundamental chamber. An element of W is called **elliptic** if it is not conjugate to an element of some proper W_T. Coxeter elements are elliptic:

3.1. Lemma. *Suppose w to be a product $s_m \ldots s_1$ of distinct reflections with $m \le n$. Then $w(v) = v$ if and only if $s_i(v) = v$ for all $i \le m$.*

Proof. By induction on m. The case $m = 1$ is a tautology. Otherwise, if $w(v) = v$ then

$$s_{m-1} \ldots s_1(v) = s_m(v) \, .$$

The right hand side is $v - 2(a_1 \bullet v)a_m$, while the left hand side is v plus a linear combination of the a_k with $k < m$. This is a contradiction unless $a_m \bullet v = 0$ and $s_{m-1} \ldots s_1(v) = v$. Apply the induction hypothesis. ∎

Therefore:

3.2. Proposition. *No Coxeter element has eigenvalue* 1.

What we shall next see, following [Steinberg:1985] is that there is a close relationship between the eigenvalues and eigenvectors of γ and those of the matrix $\mathfrak{A} = (a_i \bullet a_j)$. This relationship is best accounted for by the following result, which seems to have been first observed by Steinberg (see p. 591 of [Steinberg:1985]). There is also a version in [Berman et al.:1989] which is significantly more general than this one.

In the next result, suppose that we have renumbered the nodes of the Coxter graph so that those indexed by $[1, r]$ have even parity and those indexed by $[r + 1, n]$ have odd parity. Choose as basis the a_i, and suppose

$$\begin{aligned} x &= s_1 \ldots s_r \\ y &= s_{r+1} \ldots s_n \\ \gamma &= xy \end{aligned}$$

expressed as matrices in this coordinate system.

3.3. Lemma. *In these circumstances*

$$2I + \gamma + \gamma^{-1} = 4(I - \mathfrak{A})^2 \, .$$

Proof. In this coordinate system, the matrix $\mathfrak{A}$ is of the form

$$\mathfrak{A} = \begin{bmatrix} I & X \\ {}^tX & I \end{bmatrix} .$$

Then for $i \le r$

$$s_i(a_j) = \begin{cases} -a_i & i = j \\ a_j & i \ne j \le r \\ a_j - 2\mathfrak{A}_{i,j} a_i & r < j, \end{cases}$$

and something similar when $i > r$. Thus

$$\begin{aligned} x(a_j) &= \begin{cases} -a_j & j \le r \\ a_j - 2 \sum_{k \le r} \mathfrak{A}_{k,j} a_k & r < j \end{cases} \\ y(a_j) &= \begin{cases} a_j - 2 \sum_{k > r} \mathfrak{A}_{k,j} a_k & j \le r \\ -a_j & r < j. \end{cases} \end{aligned} \quad .$$

In terms of matrices

$$
\begin{aligned}
x &= \begin{bmatrix} -I & -2X \\ 0 & I \end{bmatrix} \\
y &= \begin{bmatrix} I & 0 \\ -2\,{}^tX & -I \end{bmatrix} \\
[x+y](a_j) &= -2\sum_k \mathfrak{A}_{k,j} a_k + 2a_j \\
x+y &= 2(I-\mathfrak{A}) \\
(x+y)^2 &= 2I + \gamma + \gamma^{-1} = 4(I-\mathfrak{A})^2 \,.
\end{aligned}
$$

These equations have been derived for a special numbering of the nodes and corresponding choice of distinguished γ. But they remain valid for any ordering, as long as x, y, and γ are chosen suitably.

There are a number of consequences. The most immediate is that γ commutes with $(\mathfrak{A}-I)^2$. Hence every eigenspace of $(\mathfrak{A}-I)^2$ is invariant under γ, and will therefore decompose into eigenspaces of γ. Since γ has finite order, all eigenvalues are of absolute value 1. Hence more precisely, if $\gamma(v) = sv$ then $4(\mathfrak{A}-I)^2(v) = (2+s+\overline{s})v$.

If $s = e^{i\theta}$ then $2+s+\overline{s} = 2+2\cos\theta$, and will equal 0 if and only if $s=-1$. I have thus proved:

3.4. Proposition. *Suppose γ to be a distinguished Coxeter element. The subspace on which $\gamma = -I$ is the same as the null space of $I-\mathfrak{A}$.*

This matches what we have seen in the picture of the icosahedron, and can see in a diagram of any of the regular polyhedra. The matrix $\mathfrak{A}$ for one of these is of the form

$$
\mathfrak{A} = \begin{bmatrix} 1 & a & 0 \\ a & 1 & b \\ 0 & b & 1 \end{bmatrix} ,
$$

and such a matrix always has 1 as an eigenvalue. This matrix need not be that associated to a Coxeter group, and the literature ([Steinberg:1959], [Berman et al.:1989], [Coleman:1989]) deals with a much more general situation concerning the product of reflections in the walls of very general acute simplices.

3.5. Lemma. *The eigenvalues of $I-\mathfrak{A}$ and of $\mathfrak{A}-I$ are the same.*

Proof. If $T = \mathfrak{A}-I$, then $T_{i,i}=0$ for all i by definition of the inner product, and for any $k>1$ the (i,j)-th entry of T^k is

$$
T^k_{i,j} = \sum_{1\le i_2,\ldots,i_k\le n} T_{i,i_2}\cdots T_{i_k,j} \,.
$$

Let $i=i_1$, $j=i_{k+1}$. The term $T_{i,i_2}\cdots T_{i_k,j}$ will be non-zero if and only if (a) we never have $i_k = i_{k+1}$ and (b) there is a path in the Coxeter graph with edges (i_k, i_{k+1}) for all k. But there are no cycles of odd length in the Coxeter graph, so all diagonal entries of T^k vanish if k is odd. Hence the trace of T^k also vanishes for all odd k. This implies that $\operatorname{trace} T^k = \operatorname{trace}(-T)^k$ for all k, and by the Newton identities relating the power sums $\sum x_i^k$ to the symmetric functions this implies that T and $-T$ have the same characteristic polynomial.

3.6. Corollary. *If n is odd, the matrix $\mathfrak{A}$ has 1 as an eigenvalue.*

Proof. Because under this assumption $\det(\mathfrak{A}-I) = -\det(I-\mathfrak{A})$, $\det(I-\mathfrak{A})=0$ and hence $I-\mathfrak{A}$ has 0 as an eigenvalue.

Proposition 3.2 implies that if n is odd -1 must appear as an eigenvalue of γ with odd multiplicity, while if n is even it must occur with even multiplicity. A case-by-case examination shows that multiplicity greater than one occurs only for groups of type D_{2m}, when it is two.

Now to consider the other, strictly complex, eigenvalues of γ. First of all, because the matrix of γ is real, these eigenvalues appear in pairs of distinct conjugate numbers, and because γ has finite order these pairs are of the form $e^{\pm i\theta}$. If v is an eigenvector for s then by the previous Lemma we have

$$4(I - \mathfrak{A})^2(v) = (2 + s + \overline{s})v$$

so that v is also an eigenvector for $4(I - \mathfrak{A})^2$. But then $\overline{v} \neq v$ is an eigenvector for $\overline{s} \neq s$, and it is also an eigenvector for $4(I-\mathfrak{A})^2$, with the same eigenvalue $2+s+\overline{s}$. Hence if V_s is the eigenspace for s then $V_s + V_{\overline{s}}$ is the eigenspace of $(I-\mathfrak{A})^2$ for $2 + s + \overline{s}$. This space is stable under $I - \mathfrak{A}$, since it commutes with $(I - \mathfrak{A})^2$. If $s = \zeta^2$, then $2 + s + \overline{s} = (\zeta + \overline{\zeta})^2$, and $\pm(\zeta + \overline{\zeta})/2$ are the corresponding eigenvalues of $I - \mathfrak{A}$.

To summarize briefly:

3.7. Proposition. *If γ is a Coxeter element then there is a bijection, with multiplicities, of the pairs of conjugate complex eigenvalues of γ and $\mathfrak{A}$, $e^{\pm i\theta}$ matching with $1 \pm \cos(\theta/2)$. The sum $V_s \oplus V_{\overline{s}}$ of eigenspaces for γ is also the sum of corresponding eigenspaces for $\mathfrak{A}$.*

There are several versions of this match, even in situations having nothing to do with a Coxeter group. This is already true to some extent in [Steinberg:1959] and [Berman et al.:1989], but the most general treatment is apparently to be found in [Coleman:1989].

So now we know that if $\lambda = 2 + 2\cos\theta$ is an eigenvalue for $4(I - \mathfrak{A})^2$ then the corresponding eigenspace decomposes into a direct sum of real planes on which γ acts by rotation through θ.

3.8. Lemma. *If γ acts on a two-dimensional plane through rotation by $\theta \neq \pi$, the plane is taken into itself by x and y which act on it by reflections in lines with angle $\theta/2$ between them.*

Proof. Since

$$xy + yx = \gamma + \gamma^{-1},$$

x and y both commute with $\gamma + \gamma^{-1}$, and the plane is stable under both of them. So we have $\gamma = xy$ with $x^2 = y^2 = 1$. If $x = \pm I$, then $\pm\gamma = y$ and $\gamma^2 = I$, so $\theta = \pi$. Otherwise, the eigenvalues of x are ± 1, and x is a reflection, say in line ℓ. But then it is easy to see that y is a reflection in a line at angle $\pm\theta/2$ with respect to ℓ.

In general, eigenvalues of γ and $\mathfrak{A}$ may occur with multiplicity greater then one. Suppose, however, that $1 - \cos(\theta_0/2)$ is the smallest eigenvalue of $\mathfrak{A}$. (Or, equivalently, that θ_0 is the smallest of the $\theta > 0$ such that $e^{i\theta}$ is an eigenvalue of γ.) This eigenvalue cannot be 0 since γ does not have 1 as an eigenvalue. In this case, something special happens:

3.9. Lemma. *The eigenspace of $\mathfrak{A}$ for eigenvalue $c = 1 - \cos(\theta_0/2)$ has dimension one.*

Proof. This c is the smallest eigenvalue of $\mathfrak{A}$, and all eigenvalues are positive. Therefore $\lambda = 1/c > 0$ is the eigenvalue largest in magnitude of $\mathfrak{A}^{-1}$. A careful analysis of Gauss elimination shows that $\mathfrak{A}^{-1}$ has only non-negative entries, and because the Coxeter graph is connected the matrix $\mathfrak{A}^{-k}$ has all positive entries for $k \gg 0$. Therefore we are reduced to a well known theorem of Perron-Frobenius. The proof is signifiacntly simpler in our circumstances, and I include it here.

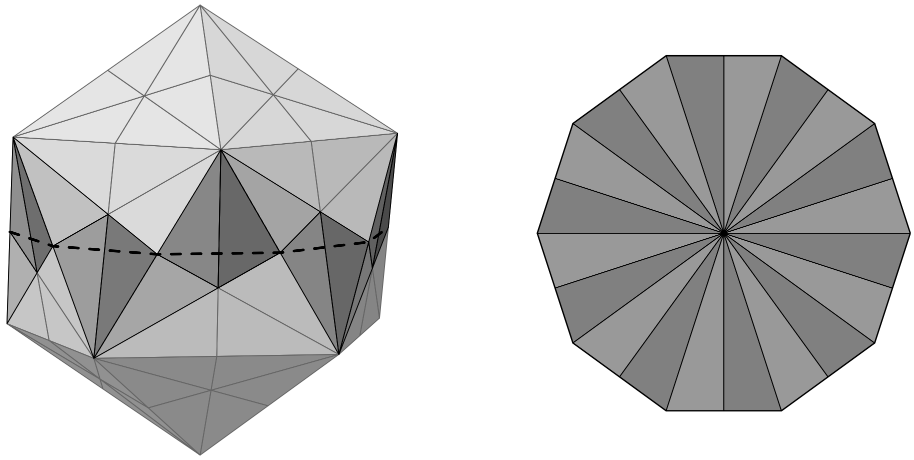
3.10. Lemma. *If M is a real symmetric matrix with positive entries, its largest eigenvalue is simple.*

Proof. For $c \gg 0$ the matrix $cI + M$ will have all eigenvalues positive, and will still have all positive entries. Its eigenvectors will be the same as those of M, so we may assume M to be positive definite. We may also scale M and assume λ, its largest eigenvalue, to be 1. If v is any non-zero vector the vectors $M^k(v)$ will have as limit an eigenvector of eigenvalue 1. If v has non-negative entries, all of these for $k \geq 1$ will be positive and bounded away from 0, so there exists an eigenvector v with all positive coordinates. If u is a linearly independent eigenvector, the plane through u and v will intersect the positive quadrant in an open planar cone. But M will take the boundary of the cone into its interior, contradicting the assumption that M acts trivially on u and v.

In practice, it is very easy to find good approximations to this eigenvalue and its eigenvector. Start with a generic vector v, and keep on repeating until satisfied: (1) replace v by Mv; (2) replace v by $v/\|v\|$.

So now let v be an eigenvector of $\mathfrak{A}$ with eigenvalue $1-\lambda$ and positive entries, u a perpendicular vector with eigenvalue $1+\lambda$. The Coxeter element γ acts by rotation on the plane spanned by u and v.

3.11. Theorem. *In the circumstances just described, the Coxeter element γ acts on the plane spanned by u and v as rotation through $2\pi/h$, where h is the order of γ.*



The involutions x and y act as reflections on this plane. If we decompose v into a sum of two components v_x fixed by x and v_y fixed by y, each will lie in the interior of a face of the fundamental chamber in which v lies. The region of the plane between the rays spanned by v_x and v_y will lie entirely inside the chamber, which implies that in acting on this plane γ has no fixed points, and that $\theta = 2\pi/h$.

4. Geometry of the Coxeter plane

Let

$$\begin{aligned} \theta &= 2\pi/h \\ c &= \cos\theta_0 \;\; s = \sin\theta_0 \\ \rho &= \cos(\theta_0/2), \;\; \sigma = \sin(\theta_0/2) \end{aligned}$$

so that

$$\begin{aligned} c &= 2\rho^2 - 1, \;\; \rho = \sqrt{(1+c)/2} \\ s &= 2\rho\sigma\,. \end{aligned}$$

The plane Π is spanned by eigenvectors of the symmetric matrix $\mathfrak{A}$ with eigenvalues $1 \pm \rho$. These can be found easily by standard techniques.

4.1. Proposition. *Given $\mathfrak{A}$, the reflections x and y are uniquely determined.*

Proof. This can be seen by going backwards. Choose a coordinate system on Π so that x amounts to reflection in the x-axis. Then

$$
\begin{aligned}
\gamma &= \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\
x &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
y &= \begin{bmatrix} c & -s \\ -s & -c \end{bmatrix} \\
I - \mathfrak{A} &= (1/2)(x + y) \\
&= \frac{1}{2} \begin{bmatrix} 1 + c & -s \\ -s & -(1 + c) \end{bmatrix} \\
&= \rho \begin{bmatrix} \rho & -\sigma \\ -\sigma & -\rho \end{bmatrix} .
\end{aligned}
$$

The vectors fixed by y are the multiples of (ρ, σ). The eigenvalues of $I - \mathfrak{A}$ are (as expected) $\pm\rho$, and its eigenvectors

$$
v_\pm = \begin{bmatrix} \sigma \\ \rho \pm 1 \end{bmatrix} ,
$$

which form a basis of Π. The vectors $(1, 0)$ and (ρ, σ) may be expressed easily as explicit linear combinations of $v_\pm$.

Let p, q be vectors in Π fixed by x, y and spanning an acute cone. Let C_Π be the interior of that cone. Since $x = s_1 \ldots s_r$ and all these s_i commute, the vector p lies on the intersection of all the hyperplanes $a_i = 0$ for $1 \le i \le r$. Similarly for y. One may choose p, q to lie on complementary walls of the fundamental chamber C, and the region C_Π is therefore the intersection of Π with C. Since Π contains points of C, it is not contained in any root hyperplane. The orbits under γ of the h lines through p, q are thus the intersections of the root hyperplanes with Π. Explicitly, we solve

$$
\begin{bmatrix} \sigma & \sigma \\ \rho + 1 & \rho - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = v
$$

for $v = (1, 0)$, (ρ, σ).

5. Root projections

Suppose that we are working with an integral root system. The reflection lines in the Coxeter plane are perpendicular to the projections of the roots themselves. The projections are easily calculated. Let $v_\pm$ be the eigenvectors of the maximum and minimum eigenvalues of $\mathfrak{A}$. Then

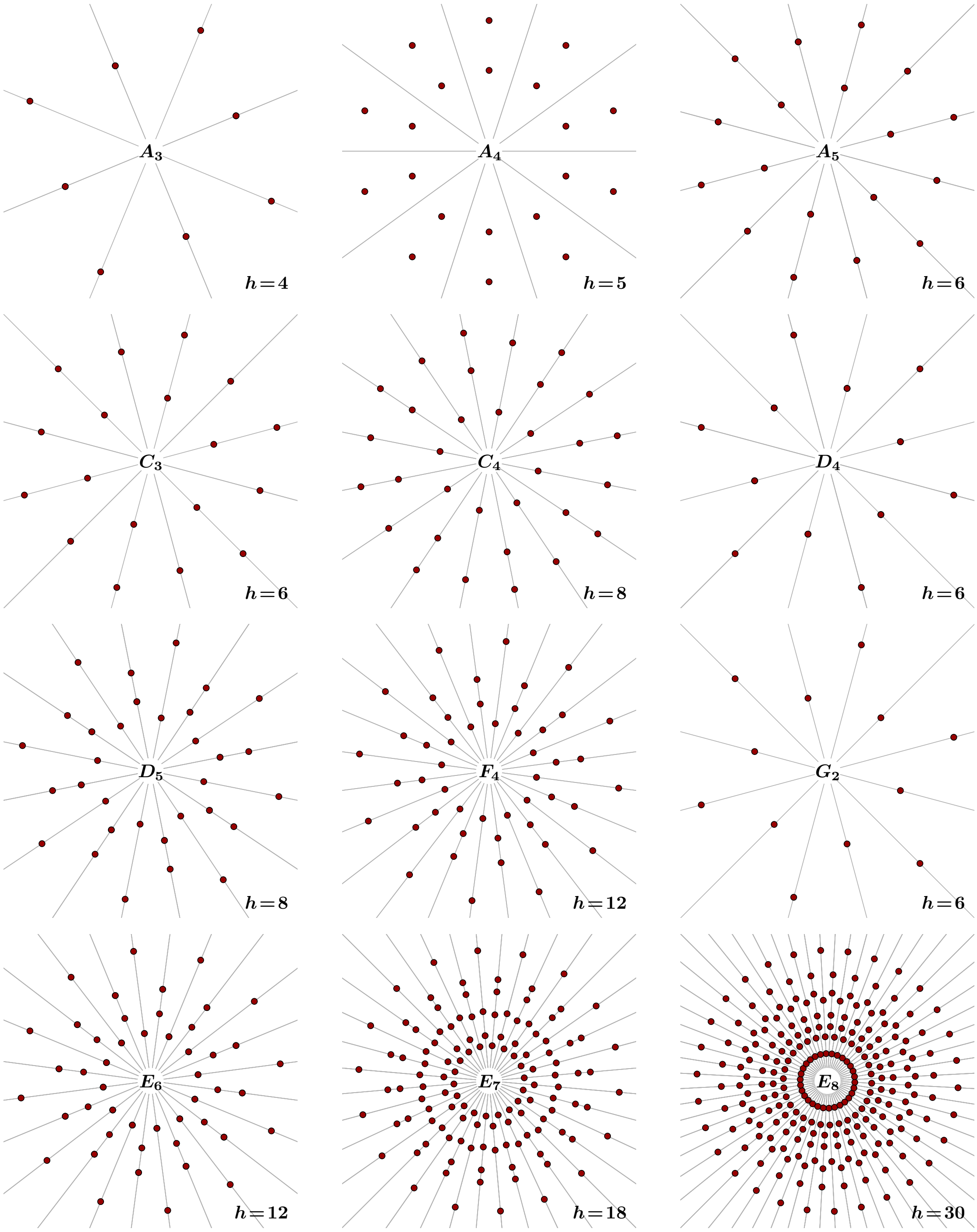
$$
{}^t v_+ \bullet v_- = 0
$$

and we must normalize them with respect to the W-invariant quadratic form

$$
{}^t u \, \mathfrak{A} \, u .
$$

Here are sample diagrams one gets:

DIAGRAMS.


A_3
$h=4$
A_4
$h=5$
A_5
$h=6$
C_3
$h=6$
C_4
$h=8$
D_4
$h=6$
D_5
$h=8$
F_4
$h=12$
G_2
$h=6$
E_6
$h=12$
E_7
$h=18$
E_8
$h=30$

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