Essays on representations of real groups

The theorem of Dixmier & Malliavin

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If (π, V) is a continuous representation of the Lie group G on a Fréchet space V, the subspace $V^{(1)}$ of differentiable vectors is that of all v such that

$$\pi(x)v = \frac{d}{dt}\pi\big(\exp(tx)\big)v\Big|_{t=0}$$

exists for all x in the Lie algebra $\mathfrak{g}$. Inductively, v lies in $V^{(n+1)}$ if it lies in $V^{(1)}$ and $\pi(x)v$ lies in $V^{(n)}$ for all x in $\mathfrak{g}$. The subspace of smooth vectors is

$$V^\infty = \bigcap V^{(n)} .$$

It is stable under both G and the universal enveloping algebra $U(\mathfrak{g})$. It is itself a Fréchet space with the semi-norms $\big\|\pi(X)v\big\|_\rho$ for X in $U(\mathfrak{g})$ and ρ a semi-norm of V. The restriction of π to V^∞ is also a continuous representation of G.

Fix a left-invariant Haar measure $dx = d_\ell x$ on G. For f in $C_c^\infty(G)$, let

$$\pi(f) = \int_G f(x)\pi(x)\,dx .$$

We have

$$\pi(y)\pi(f) = \int_G f(x)\pi(yx)\,dx = \int_G f(y^{-1}z)\pi(z)\,dz = \pi(L_y f) ,$$

where L is the left regular representation of G. From this it can be deduced that for any v in V and f in $C_c^\infty(G)$ the vector $\pi(f)$ is smooth, and more precisely that if X lies in $U(\mathfrak{g})$ then $\pi(X)\pi(f)v = \pi(L_X f)v$. This implies that V^∞ is dense in V, since if $\{f_n\}$ is a Dirac sequence on G then $\pi(f_n)v \to v$. The subspace of V^∞ spanned by the $\pi(f)v$ with f in $C_c^\infty(G)$ is called the Gårding subspace of V.

It is relatively easy to show that a smooth vector may be expressed as a linear combination of $\pi(f)v$ with f in $C_c^m(G)$ for arbitrarily high m, as explained in [Cartier:1974]. I'll say something about this in the first section. It is considerably more difficult to see that if V is a Fréchet space then the smooth vectors and the Gårding subspace coincide. This remarkable result was proved in [Dixmier-Malliavin:1978].

Their main result, which is what this essay is concerned with, is this:

Theorem (Dixmier-Malliavin) *Suppose G to be a Lie group, (π, V) a continuous representation of G on a Fréchet space. Every smooth vector v in V may be represented as a finite linear combination*

$$v = \sum \pi(f_k)v_k$$

with each f_k in $C_c^\infty(G)$ and each v_k in V^∞.

After an introductory section, I'll follow closely the exposition of [Dixmier-Malliavin:1978], at least in substantial matters. But their paper is quite condensed and their logic is intricate, and I have tried to come up with a more relaxed account. In one place, I'll correct what seems to be a minor error in their treatment. And in the third section I'll pose a problem of general interest about infinite products that is quite easy to state, but not obviously so easy to solve.

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1. Introduction

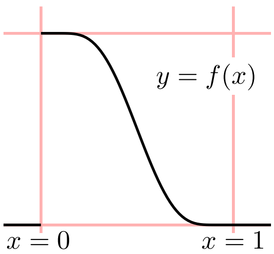
I'll summarize what was known before the work of Dixmier and Malliavin. The basic fact is one about Euclidean space $\mathbb{R}^n$.

1.1. Proposition. *For each $m \geq 0$ we can express*

$$\delta_0 = \sum_{k \leq m} \partial^k f_i / \partial x^k$$

where each f_i lies in $C_c^m(\mathbb{R}^n)$.

Proof. This is relatively elementary. Suppose at first that $n = 1$.



Let $g(x)$ be a smooth non-negative function on $\mathbb{R}$ with support on $[0, 1]$, such that

$$\int_0^1 g(x)\, dx = 1$$

and set

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \displaystyle\int_x^\infty g(x)\, dx & \text{for } x > 0. \end{cases}$$

The function $f(x)$ has a jump discontinuity at $x = 0$, and more precisely as a distribution it satisfies the equation

$$Df = \delta_0 - g(x) \quad \delta_0 = Df + g\,,$$

where $D = d/dx$. We can continue on to higher derivatives. The trick is to consider the functions $x^m f$ instead of f. For example: (a) xf is C^0 and

$$\begin{aligned} Dxf &= f + x\,Df \\ &= f + x\delta_0 - xg(x) \\ &= f - g_{1,1} \quad \text{(say)} \\ D^2xf &= Df - D(xg) \\ &= \delta_0 - (g + D(xg)) \\ &= \delta_0 - g_{2,1} \end{aligned}$$

since $x\delta_0 = 0$. (b) The function $x^2 f$ is C^1 and

$$\begin{aligned} Dx^2 f &= 2xf + x^2\, Df \\ &= 2xf + x^2\delta_0 - x^2 g(x) \\ &= 2xf - x^2 g(x) \\ &= 2xf - g_{1,2} \\ D^2x^2 f &= 2f + 2xDf - Dg_{1,2} \\ &= 2f - (2xg + Dg_{1,2}) \\ &= 2f - g_{2,2} \\ D^3x^2 f &= 2Df - D() \\ &= 2\delta_0 - (2g + Dg_{2,2})\,. \\ &= 2\delta_0 - g_{3,2}\,. \end{aligned}$$

Here each $g_{k,m}$ is smooth. Thus we now set

$$m^{[k]} = \begin{cases} 1 & \text{if } k = 0 \\ m(m-1)\ldots(m-k+1) & k > 0. \end{cases}$$

and define smooth functions $g_{k,m}$ inductively:

$$\begin{aligned} g_{\,0,m} &= 0 \\ g_{\,k+1,m} &= m^{[k]}x^{m-k}g + Dg_{k,m} \end{aligned}$$

One can now verify by induction on k for each fixed m that

$$\begin{aligned} D^k x^m f &= m^{[k]}x^{m-k}f + g_{\,k,m} \quad (k \le m) \\ D^{m+1}x^m f &= m!\,\delta_0 + g_{\,m+1,m}\,. \end{aligned}$$

This proves the Proposition in dimension one. In higher dimensions, one uses this equation together with the fact that δ_0 can be interpreted as the product of distributions $\delta_{x_i=0}$ with support on the coordinate hyperplanes. In dimension two, for example,

$$\frac{\partial^2}{\partial x\partial y}\big(f(x)f(y)\big) - \frac{\partial}{\partial x}\big(f(x)g(y)\big) - \frac{\partial}{\partial y}\big(g(x)f(y)\big) + g(x)g(y) = \delta_0\,. \ \blacksquare$$

If G is abelian, then the following result is immediate. Otherwise, a mildly technical argument that I won't reproduce here (and which can be extracted from the argument in the last section of this essay) implies:

1.2. Corollary. *If (π, V) is a continuous representation of the Lie group G and v lies in V^∞ then for every $m \ge 0$ one can write*

$$v = \sum \pi(f_i)v_i$$

with f_i in $C_c^{(m)}(G)$ and v_i in V^∞.

This has been adequate for many purposes in representation theory, but the result of Dixmier and Malliavin is a drastic improvement. Its proof is very intricate, a real *tour de force*. Much of what is to come is therefore rather technical. I'll try to motivate it here, more or less by going backwards.

First of all, the argument in the last section of this essay reduces the question to one about distributions with compact support in dimension one. This in turn can be reduced to one about tempered distributions in dimension one. There are some technical aspects to this reduction I won't explain here, partly involved in the reduction from compact support to tempered, but very roughly the problem about tempered distributions is this: *can we find a series* (c_m) *and a function* ψ *in* $\mathcal{S}(\mathbb{R})$ *such that*

$$\sum_0^\infty a_m \psi^{(m)} = \delta_0\,?$$

At first sight this seems a very unfamiliar problem, but if we apply the Fourier transform to this equation, at least formally, we get the equivalent problem of finding a function χ in $\mathcal{S}(\mathbb{R})$ such that

$$\Big(\sum (2\pi i)^m a_m x^m\Big)\chi(x) = 1 \;\text{ or }\; \chi(x) = \frac{1}{\sum (2\pi i)^m a_m x^m}\,.$$

This suggests setting

$$(2\pi)^m a_m = \begin{cases} 0 & \text{if } m \text{ is odd} \\ (-1)^\ell \alpha_\ell & \text{if } m = 2\ell, \end{cases}$$

with $\alpha_\ell > 0$. Then the formula becomes (still formally)

$$\chi(x) = \frac{1}{\sum \alpha_\ell x^{2\ell}}\,,$$

which implies that at least $\chi(x)$ is of rapid decrease at ∞. Proving that its derivatives are also of rapid decrease, and satisfying various other technical conditions on the coefficients α_ℓ, is more difficult. What Dixmier and Malliavin do is search for functions with infinite product expansions

$$\varphi(z) = \sum \alpha_\ell x^{2\ell} = \prod \left(1 + \frac{x^2}{\lambda_\ell^2}\right).$$

Under mild conditions on the λ_ℓ this will define an entire function with simple zeroes $z = \pm\lambda_\ell\, i$. Thus χ will have simple poles at those same values of z, and under a very restrictive condition on the λ_ℓ its inverse Fourier transform will have an expansion

$$\psi(y) = \sum \frac{e^{-2\pi\lambda_\ell y}}{\varphi'(\lambda_\ell i)}\,,$$

which turns out to be all we need. A very crude model, as we'll see, is

$$\varphi(z) = \frac{\sinh(z)}{z}$$

with

$$\frac{1}{\varphi(x)} = 2|x|e^{-|x|}(1 + e^{-2|x|} + e^{-4|x|} + \cdots)$$

for $x \neq 0$.

I assume from now on throughout this essay that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ *is a monotonic sequence of positive numbers such that*

$$\sum \frac{1}{\lambda_m^2} < \infty\,.$$

In particular, the λ_m are unbounded. From standard results about infinite products it follows that:

1.3. Proposition. *The infinite product*

$$\varphi_\lambda(z) = \left(1 + \frac{z^2}{\lambda_1^2}\right)\left(1 + \frac{z^2}{\lambda_2^2}\right)\left(1 + \frac{z^2}{\lambda_3^2}\right)\ldots$$

defines an entire function of z *whose zeroes are simple and located at* $\pm\lambda_m i$.

One well known example of Proposition 1.3 is

$$\frac{\sinh(z)}{z} = \frac{e^z - e^{-z}}{2z} = \left(1 + \frac{z^2}{\pi^2}\right)\left(1 + \frac{z^2}{4\pi^2}\right)\left(1 + \frac{z^2}{9\pi^2}\right)\ldots$$

2. Infinite products and Schwartz functions

In this section and the next, let $G = \mathbb{R}$. I recall that the **Schwartz space** $\mathcal{S}(\mathbb{R})$ is the space of all smooth functions $f(x)$ on $\mathbb{R}$ all of whose derivatives are $O(1/|x|^N)$ at infinity for all N.

This section will be concerned with a very general question raised implicitly by [Dixmier-Malliavin:1978], and answered by them for one very special family of cases. Suppose, as earlier, that

$$\varphi_\lambda(z) = \left(1 + \frac{z^2}{\lambda_1^2}\right)\left(1 + \frac{z^2}{\lambda_2^2}\right)\left(1 + \frac{z^2}{\lambda_3^2}\right)\dots$$

The inverse

$$\chi_\lambda(x) = \frac{1}{\varphi_\lambda(x)}$$

is meromorphic on $\mathbb{C}$ with simple poles at $\pm\lambda_m i$. The following question arises naturally:

> *The function $\varphi_\lambda(x)$ grows faster at infinity on $\mathbb{R}$ than any polynomial, and hence the restriction of χ_λ to $\mathbb{R}$ is of rapid decrease at infinity. In what circumstances is this restriction in the Schwartz space?*

Since

$$\varphi'_\lambda(z) = \sum_k \frac{2z}{\lambda_k^2} \prod_{\ell \neq k} \left(1 + \frac{z^2}{\lambda_\ell^2}\right)$$

the residue at $\lambda_m i$ is $1/\varphi'_\lambda(i\lambda_j)$, where

$$\varphi'_\lambda(i\lambda_j) = \frac{2i}{\lambda_j} \prod_{\ell \neq j} \left(1 - \frac{\lambda_j^2}{\lambda_\ell^2}\right).$$

For example, we have already seen that the function $x/\sinh x$ has a product expansion of the type we are considering. It lies in $\mathcal{S}(\mathbb{R})$, since it is even and we have a converging expansion for $|x| \neq 0$:

$$\frac{x}{\sinh(x)} = \frac{2x}{e^x - e^{-x}} = 2|x|e^{-|x|}\big(1 + e^{-2|x|} + e^{-4|x|} + \cdots\big).$$

Hence all of its derivatives are of essentially exponential decrease at ∞.

I do not know of any really satisfactory conditions on the sequence λ_m that guarantee that χ_λ will lie in $\mathcal{S}(\mathbb{R})$, but it seems often to be the case. [Dixmier-Malliavin:1978] proves that $\chi_\lambda(x)$ lies in the Schwartz space for a class of sequences λ satisfying rather stringent conditions. I generalize their conditions slightly, and define a sequence λ_m to be **admissible** if $\lambda_1 \geq 1$ and $\lambda_{m+1}/\lambda_m > q > 1$. The condition on λ_1 is just a matter of convenience. One immediate consequence of admissibility is that both

$$\lambda_m \geq q^m$$
$$\lambda_{m+k}/\lambda_m \geq q^k.$$

Later on, we'll see very specific admissible sequences, chosen to satisfy various criteria. Dixmier and Malliavin assume that the λ_m are a monotonic subsequence of the positive powers of 2 (i.e. that $q = 2$), but I believe that looking at a slightly more general case is illuminating.

2.1. Theorem. *If (λ_m) is an admissible sequence then the function $\chi_\lambda(x)$ lies in $\mathcal{S}(\mathbb{R})$.*

Proof. It will be long, and parts of it of some independent interest. I'll not yet assume that λ is admissible.

Since χ_λ is of rapid decrease, its inverse Fourier transform

$$\psi_\lambda(y) = \int_{-\infty}^{\infty} \chi_\lambda(x) e^{-2\pi i x y}\, dx$$

is a well defined, smooth, and even, and in order to show that χ_λ lies in $\mathcal{S}(\mathbb{R})$ it suffices to show that ψ_λ lies in $\mathcal{S}(\mathbb{R})$. For any $t \geq 0$ not equal to one of the $\lambda_m i$ the real function $x \mapsto \chi_\lambda(x+it)$ is still of rapid decrease. We may now shift contours, picking up residues as we do so, to deduce that

$$\psi_\lambda(y) = \sum_{j=0}^{k} \frac{e^{-2\pi\lambda_j y}}{\varphi'(i\lambda_j)} + e^{-2\pi t y} \int_{-\infty}^{\infty} \chi_\lambda(x+it) e^{2\pi i x y}\, dx$$

if $\lambda_k < t = t_k < \lambda_{k+1}$. We want to be able to move t_k to infinity. In the end we'll get this result:

2.2. Proposition. *If (λ_m) is admissible, then*

(a) for $y > 0$, the Fourier transform of χ_λ may be expressed as

$$\psi_\lambda(y) = \sum_{j=0}^{\infty} \frac{e^{-2\pi\lambda_j y}}{\varphi'(i\lambda_j)}$$

(b) there exists a constant $C_\lambda > 0$ such that

$$\frac{1}{\left|\varphi'(i\lambda_m)\right|} \leq \frac{\lambda_m}{C_\lambda}$$

for every m.

It will follow from the proof of (a) that the series in question converges, and (b) strengthens this considerably.

Proof. I'll first begin the proof of (a), but as we'll see the proofs of both reduce to almost identical questions. We have

$$\chi_\lambda(x+it) = \frac{1}{\varphi_\lambda(x+it)}, \quad \varphi_\lambda(x+it) = \prod_{j\geq 1} \left(1 + \frac{(x+it)^2}{\lambda_j^2}\right).$$

I am going to separate the first factor from the others. Let's look at the norm of that factor, assuming $t \geq \lambda_1$:

$$\begin{aligned}
\left|1 + \left(\frac{x+it}{\lambda_1}\right)^2\right|^2 &= \left|1 + (s+i\tau)^2\right|^2 \quad (s = x/\lambda_1, \tau = t/\lambda_1) \\
&= \left|(1 + s^2 - \tau^2) + 2is\tau\right|^2 \\
&= (1 + s^2 - \tau^2)^2 + 4s^2\tau^2 \\
&= (s^2 - \alpha^2)^2 + 4s^2(1 + \alpha^2) \\
&= (s^2 + \alpha^2)^2 + 4s^2 \geq s^4 + 4s^2 + 1
\end{aligned}$$

if $\alpha^2 = \tau^2 - 1 \geq 0$. Thus when $t \geq \lambda_1$

$$\left|\varphi_\lambda(x+it)\right| \geq (s^4 + 4s^2 + 1) \prod_{j\geq 2} \left(1 + \frac{(x+it)^2}{\lambda_j^2}\right).$$

Since $(s^4 + 4s^2 + 1)^{-1/2}$ is integrable over $(-\infty, \infty)$ we shall be through proving (a) if we can choose the t_k suitably and find a lower bound for

$$\prod_{j\geq 2} \left(1 + \frac{(x+it_k)^2}{\lambda_j^2}\right)$$

valid for all large enough k. Following Dixmier and Malliavin, I choose

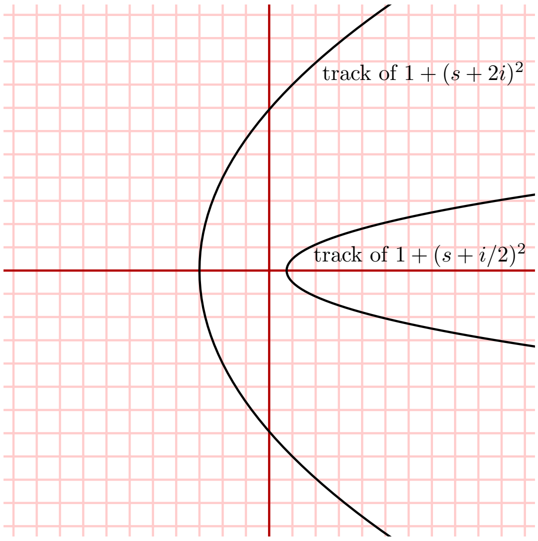
$$t_k = \sqrt{\lambda_k \lambda_{k+1}}\,.$$

Since

$$\varphi'_\lambda(i\lambda_\ell) = \frac{2i}{\lambda_\ell} \prod_{j \neq \ell} \left(1 - \frac{\lambda_\ell^2}{\lambda_j^2}\right),$$

the proof of (b) will also follow if we can find a good lower bound for some infinite products.

I first find a lower bound for some of the terms in the product. The track of $1 + (s + i\tau)^2$ as s ranges over all of $\mathbb{R}$ behaves in one of two possible ways, depending on whether $\tau > 1$ or $\tau < 1$. In the first case it swings around the origin to its left, and in the second to its right:



In either case

$$\left| 1 + \left(\frac{x + it}{\lambda_j}\right)^2 \right| \geq \left|1 - (t/\lambda_j)^2\right|.$$

To conclude the proof of the Lemma, we must find a lower bound for

$$\left(1 - \frac{\lambda_k \lambda_{k+1}}{\lambda_2^2}\right)\left(1 - \frac{\lambda_k \lambda_{k+1}}{\lambda_3^2}\right) \cdots \left(1 - \frac{\lambda_k \lambda_{k+1}}{\lambda_m^2}\right) \cdots$$

independent of k, and one for

$$\prod_{m \neq k} \left(1 - \frac{\lambda_k^2}{\lambda_m^2}\right)$$

also independent of k. The arguments in the two cases are almost identical, so I'll do just one. We shall need this rather elementary result:

2.3. Lemma. *If the a_j are a sequence of positive numbers with $\sum a_j < 1$ then*

$$\prod (1 - a_j) > 1 - (a_0 + a_1 + a_2 + \cdots).$$

Proof. By induction. The hypotheses imply that $0 < a_j < 1$. Let $p_n = \prod_{j<n}(1-a_j)$, $s_n = a_0 + a_1 + \cdots + a_{n-1}$. By assumption $p_n < 1$, $s_n < 1$. To start, $p_1 = 1 - a_0 = 1 - s_1 > 0$. For the induction step, if $p_n \geq 1 - s_n$ then

$$p_{n+1} = p_n(1 - a_n) > (1 - s_n)(1 - a_n) = 1 - s_n - a_n + s_n a_n > 1 - s_{n+1}. \quad \blacksquare$$

I divide up the terms in the infinite product into several cases, the basic dichotomy being $j \leq k$ and $j > k$, or equivalently $t_k/\lambda_j > 1$ and $t_k/\lambda_j < 1$ (with $t_k = \sqrt{\lambda_k \lambda_{k+1}}$).

The first step is to find a lower bound for the infinite tail product

$$\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+1}^2}\right)\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+2}^2}\right)\cdots\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+1}^2}\right)\cdots$$

Now for $\ell \ge 0$

$$\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+1}^2} \le \frac{1}{q^{2\ell+1}}, \quad 1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+1}^2} \ge 1-\frac{1}{q^{2\ell+1}}\,.$$

If ℓ is large enough this will be very close to 1. Then we have

$$\begin{aligned}\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+1}^2}\right)&\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+2}^2}\right)\left(1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k+\ell+3}^2}\right)\cdots\\ &> \left(1-\frac{1}{q^{2\ell+1}}\right)\left(1-\frac{1}{q^{2\ell+3}}\right)\left(1-\frac{1}{q^{2\ell+5}}\right)\cdots\\ &> 1-\left(\frac{1}{q^{2\ell+1}}+\frac{1}{q^{2\ell+3}}+\frac{1}{q^{2\ell+5}}+\cdots\right)\\ &= 1-\frac{1}{q^{2\ell+1}}\left(1-\frac{1}{q^2}\right)^{-1}.\end{aligned}$$

So if we set

$$Q_\ell = \left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^3}\right)\cdots\left(1-\frac{1}{q^{2\ell-1}}\right)$$

he whole tail product will be greater than $Q_\ell/2$.

Something similar happens for $m \le k$. Here

$$\frac{\lambda_k\lambda_{k+1}}{\lambda_{k-\ell}^2} \ge q^{2\ell+1}, \quad 1-\frac{\lambda_k\lambda_{k+1}}{\lambda_{k-\ell}^2} \le 1-q^{2\ell+1}\,.$$

If we choose ℓ large enough so $q^{2\ell+1} \ge 2$, then this is ≤ 1, hence ≥ 1 in absolute value. All in all, as long as $k \ge \ell+1$ we get a lower bound on the magnitude of the product

$$\prod_{j=2}^{\infty}\left|1-\frac{t_k^2}{\lambda_j^2}\right|$$

that is independent of k. This concludes the proof of Proposition 2.2, ▯

To conclude the proof of Theorem 2.1, I must show that all the functions $x^m d^n\psi_\lambda/dx^n$ are bounded for $y > 0$. I shall prove, among other things, that the derivatives of ψ_λ are computed by taking derivatives of the terms in the series.

Formally we have

$$y^m\psi_\lambda^{(n)}(y) = \sum_{j=0}^{\infty}(-2\pi\lambda_j)^n\,\frac{y^m e^{-2\pi\lambda_j y}}{\varphi'(i\lambda_j)}\,,$$

and this series for the derivatives converges according to part (b) of Proposition 2.2.

For the series for $y^m\psi_\lambda^{(n)}(y)$ we therefore have the crude but adequate estimate

$$\left|\sum_{j=0}^{\infty}(-2\pi\lambda_j)^n\,\frac{y^m e^{-2\pi\lambda_j y}}{\varphi'(i\lambda_j)}\right| \le \frac{(2\pi)^n}{C}\sum_{j=0}^{\infty}\lambda_j^{n+1}y^m e^{-2\pi\lambda_j y}\,.$$

The function $y \mapsto y^m e^{2\pi\lambda y}$ in the range $y > 0$ has its maximum value at $m/2\pi\lambda$, and is there equal to $(m/2\pi\lambda e)^m$. The series is therefore dominated by

$$\left(\frac{m}{2\pi e}\right)^m\sum\lambda_j^{n+1-m}\,.$$

Since this converges for $n+1-m \le -2$ or $m \ge n+3$, the series for $y^m\psi_\lambda^{(n)}(y)$ converges uniformly in the region $y > 0$ and the function is bounded there. This concludes the proof of Theorem 2.1. ▯

3. Interlude: a curious question

The strategy applied here, of bounding each factor in an infinite product, is very special to admissible sequences λ_m. Consider instead the product expansion

$$\varphi(z) = \sinh(z)/z = \prod \left(1 + \frac{z^2}{\pi^2 m^2}\right), \quad \chi(z) = \frac{z}{\sinh(z)} .$$

In this case with $t_k = \pi(k + 1/2)$

$$\frac{1}{\varphi(x + it_k)} = \frac{2(x + \pi(k + 1/2)i)}{(-1)^k i(e^x + e^{-x})}$$

so that once again we can pass the integral terms

$$e^{-2\pi ky} \int_{\mathbb{R}} \chi(x + it_k) e^{2\pi ixy}\, dx$$

off to $t_k = \infty$. Also

$$\varphi'(\pi mi) = \frac{\cosh(\pi mi)}{\pi mi} = \frac{(-1)^m}{\pi mi}$$

so one can use the series

$$\psi(y) = \sum_{m=1} (-1)^m (\pi mi) e^{-\pi my}$$

to show that $\chi(x)$ is in the Schwartz space, by explicit calculation. But this is not at all easy to tell from the infinite product expansion. There is thus some evidence that Theorem 2.1 and even Proposition 2.2 are true in very general circumstances, but it is not easy to see a common factor in the proofs of the cases in which they are known to be true.

4. The main lemma

The group $\mathbb{R}$ acts on $\mathcal{S}(\mathbb{R})$ by translation:

$$L_x \varphi(y) = \varphi(y - x)$$

and this gives rise to an action by functions in $C_c^\infty(\mathbb{R})$: The multiplicative group also acts:

$$\mu_c f(x) = f(x/c) .$$

$$L_f \varphi = \int_{\mathbb{R}} f(y) L_y \varphi \, dy, \quad [L_f \varphi](y) = \int_R f(x) \varphi(y - x)\, dx = [f * \varphi](y) .$$

This is also

$$\int_{\mathbb{R}} f(x)\, \mu_{-1}[L_y \varphi](x)\, dx$$

and so the formula for convolution makes sense if f is a tempered distribution.

Let $\widehat{S}(\mathbb{R})$ be the topological dual of $\mathcal{S}(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$. The key lemma in the proof of the theorem of Dixmier-Malliavin is this:

4.1. Lemma. (Main Lemma) *Given any sequence of positive constants A_m ($m \ge 1$), there exists a sequence of positive constants α_m with $\alpha_0 = 1$ and $\alpha_m \le A_m$ for $m \ge 1$, as well as a function ψ in $\mathcal{S}(\mathbb{R})$, such that*

$$\sum_{j=0}^{m} (-1)^j \alpha_j \delta^{(2j)} * \psi \longrightarrow \delta$$

in $\widehat{S}(\mathbb{R})$ as $m \to \infty$.

The function ψ will be chosen as ψ_λ for an admissible sequence λ_m that will arise in the course of the proof, depending on the A_m. From now on, following [Dixmier-Malliavin:1978], I assume that $q = 2$ for all admissible sequences, so $\lambda_m = 2^{n_m}$ with $n_{m+1} > n_m$.

We can derive a useful bound from the series for ψ_λ. As we have seen, this gives us for $y \geq 1$

$$\left|\psi_\lambda^{(n)}(y)\right| \leq C_\lambda (2\pi)^n \sum_{k=0}^{\infty} \lambda_k^{n+1} e^{-2\pi\lambda_k} ,$$

where C_λ is chosen so

$$\left|\frac{1}{\varphi'_\lambda(\lambda_k i)}\right| \leq C_\lambda \lambda_k$$

for all k. Let $F(\lambda) = \lambda^{n+1} e^{-2\pi\lambda}$. Since

$$F'(\lambda) = (n+1)\lambda^n e^{-2\pi\lambda} - (2\pi) e^{-2\pi\lambda} = (n+1-2\pi\lambda)\lambda^{n-1} e^{-2\pi\lambda} ,$$

it is monotonic increasing from $\lambda = 0$ to $\nu_n = (n+1)/2\pi$, where it takes a maximum value μ_n, and from there on monotonic decreasing. We now have an estimate

$$(2\pi)^{-n} C^{-1} \left|y^m \psi_\lambda^{(n)}(y)\right| \leq \sum_{\lambda_k \leq \nu_n} \mu_n + \sum_{\lambda_k > \nu_n} \lambda_k^{n+1} e^{-2\pi\lambda_k} .$$

Since $\lambda_k \geq 2^k$ the first sum is at most $\lfloor \log_2 \nu_n \rfloor \mu_n$. Since $\lambda_{k+1} \geq 2\lambda_k$ the second is at most

$$\sum_{k=0} (\nu_n 2^k)^{n+1} e^{-2\pi\nu_n 2^k} .$$

It is in this last step that [Dixmier-Malliavin:1978] seems to err, apparently assuming $\lambda^{n+1} e^{-2\pi\lambda}$ to be monotonic throughout. No matter, in any case their final claim is true:

4.2. Lemma. *For each n there exists a number M_n such that*

$$\left|\psi_\lambda^{(n)}(y)\right| \leq M_n$$

for all $y \geq 1$ and all admissible sequences λ.

Now to conclude the proof of the Main Lemma. We shall choose the λ_k by induction. First of all, recall that $A_0 = 1$. For the induction step, suppose we have chosen the λ_k for $k < m$ so that in the expansion

$$\sum \alpha_k^{(m)} z^{2k} = \prod\nolimits_{k=1}^{m} \left(1 + \frac{z^2}{\lambda_k^2}\right)$$

each $\alpha_k^{(m)} < (2\pi)^k A_k$. Now choose λ_m large enough among the powers of 2 so that in the polynomial expansion of

$$\prod\nolimits_{k=0}^{m} \left(1 + \frac{z^2}{\lambda_k^2}\right) = \left(1 + \frac{z^2}{\lambda_m^2}\right) \cdot \prod\nolimits_{k=0}^{m-1} \left(1 + \frac{z^2}{\lambda_k^2}\right)$$

each coefficient $\alpha_k^{(m+1)} < (2\pi)^k A_k$. In the limit one obtains the expansion

$$\sum_0^{\infty} \alpha_k z^{2k} = \chi_\lambda(z)$$

with each $\alpha_k \le (2\pi)^k A_k$. Thus for each finite m we have

$$\Phi_m(z) = \frac{\sum_0^m \alpha_k z^{2k}}{\chi_\lambda(z)} < 1$$

and $\Phi_m \longrightarrow 1$ both as a function and as a tempered distribution (convergence in the weak topology). Taking Fourier transforms, we get

$$\sum_0^m (-1)^k \alpha_k \frac{\delta^{(2k)}}{(2\pi)^{2n}}\, \psi_\lambda \longrightarrow \delta$$

as $m \to \infty$. ▯

We can now deduce easily a variant of the main theorem of Dixmier-Malliavin. It is not particularly important, but it displays directly the basic mechanism of the proof of the theorem itself.

4.3. Proposition. *Given φ in $\mathcal{S}(\mathbb{R})$ there exist f and g in $\mathcal{S}(\mathbb{R})$ with $\varphi = f * g$.*

Proof. I start with an elementary if clumsy lemma, one which will be used also a few times later on.

4.4. Lemma. *Given a map $C_{j,k}$ from $\mathbb{N}^{1+p}$ to $\mathbb{R}$, there exists a sequence (α_j) such that*

$$\sum_j \alpha_j |C_{j,k}| < \infty$$

for every $k \in \mathbb{N}^p$.

Proof. Set $C_j^* = \sup_{m \le j, n \le (j,j,\dots,j)} |C_{m,n}|$. Here $m \le n$ in $\mathbb{N}^n$ means inequality component-wise. Thus C_j^* is a monotonically increasing sequence. Choose the α_j so that $\sum \alpha_j C_j^* < \infty$, say by setting $\alpha_j = 1/j^2 C_j^*$. For k in $\mathbb{N}^p$ let $|k|$ be the maximum value of its coordinates. Then

$$\sum_0^\infty \alpha_j |C_{j,k}| = \sum_{j<|k|} \alpha_j |C_{j,k}| + \sum_{j=|k|}^\infty \alpha_j |C_{j,k}| \le \sum_{j<|k|} \alpha_j |C_{j,k}| + \sum_{j=|k|}^\infty \alpha_j |C_j^*| < \infty\,. \text{ ▯}$$

Given φ in $\mathcal{S}(\mathbb{R})$, let $C_{j,k,\ell} = \sup |y|^k |\varphi^{(2j+\ell)}(y)|$. Choose the A_j according to the lemma, so that $\sum A_j C_{j,k,\ell} < \infty$ for all k, ℓ.

Choose ψ according to the Main Lemma so that

$$\sum (-1)^j \alpha_j \delta^{(2j)} * \psi \to \delta, \quad \alpha_j \le A_j\,.$$

Then

$$\sum (-1)^j \alpha_j \delta^{(2j)} * \psi * \varphi = \sum (-1)^j \alpha_j \delta^{(2j)} * \varphi * \psi \to \varphi$$

in $\mathcal{S}(\mathbb{R})$. Because $\alpha_j \le A_j$, the sum $\sum \alpha_j \varphi^{(2j)}$ converges to a function Φ in $\mathcal{S}(\mathbb{R})$, for any sequence $\alpha_j \le A_j$, and $\Phi * f = \varphi$. ▯

This is the simplest example of an argument we'll see repeated later on.

5. Functions of compact support on *R*

From now on I follow [Dixmier-Malliavin:1978] more closely. In this section, let $C^{-\infty}(\mathbb{R})$ be the space of distributions on $\mathbb{R}$. It is the topological dual of $C_c^\infty(\mathbb{R})$.

5.1. Lemma. *Given any sequence of positive constants A_m, there exists a sequence of positive constants $\alpha_m \le A_m$ and functions f, g in $C_c^\infty(\mathbb{R})$ such that*

$$\sum_{i=0}^{m} (-1)^i \alpha_i \delta^{(2i)} * f \to \delta + g$$

in $C^{-\infty}(\mathbb{R})$ as $m \to \infty$. The functions f and g may be given arbitrarily small support.

Proof. Let ω be a smooth, even function on $\mathbb{R}$ with support on $[-3,3]$, identically 1 on $[-2,2]$. For every sequence λ set

$$\omega_\lambda = \omega \cdot \psi_\lambda \,,$$

with ψ_λ defined as in the proof of the Main Lemma. By Lemma 4.2

$$\sup_{y \ge 1} |\omega_\lambda^{(n)}| \le D_n$$

for suitable constants D_n and all λ. Replace the constants A_m by new ones

$$B_m = \sup(A_m, 1/n^2 D_{2n}, \ldots, 1/n^2 D_{2n+n}) \,.$$

I claim that

$$\sum_0^p (-1)^j \alpha_j \frac{\delta^{(2j)}}{(2\pi)^{2n}} * \omega_\lambda \longrightarrow \delta + g$$

for some g in $C_c^\infty(\mathbb{R})$. I suffices to verify this separately on intervals $(-2,2)$, $(1,4)$, $(3,\infty)$. On the first we are just repeating the previous argument, since there $\omega_\lambda = \psi_\lambda$ and therefore

$$\sum_0^p (-1)^j \alpha_j \frac{\delta^{(2j)}}{(2\pi)^{2n}} * \omega_\lambda \longrightarrow \delta \,.$$

For $y \ge 1$ we have

$$\left| \alpha_n \frac{\delta^{(2n+p)}}{(2\pi)^{2n}}\, \omega_\lambda(y) \right| \le \alpha_n D_{2n+p} \le \frac{1}{n^2} \quad \text{if } n \ge p$$

so

$$\sum_0^p (-1)^n \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}}\, \omega_\lambda$$

has as limit on a function in $C_c^\infty([1,4])$.

Scaling, one may assume f and g in this lemma to have support in arbitrarily small intervals. ▯

6. Smooth representations

Now for the proof of the main theorem. Let G be a Lie group, (π, V) a continuous representation of G on the Fréchet space V, V^∞ the subspace of smooth vectors in V. AS I have already mentioned, it also is a Fréchet space on which G acts continuously. Fix a neighbourhood Ω of the identity of G on which the exponential map is a diffeomorphism.

6.1. Proposition. *Any v in V^∞ can be expressed as a finite sum*

$$v = \sum \pi(f_j) v_j$$

with each $f_j \in C_c^\infty(\overline{\Omega})$, $v_j \in V^\infty$.

Proof. Let (X_j) be a basis of $U(\mathfrak{g})$, x in $\mathfrak{g}$ such that $\exp tx$ lies in Ω for $t \le 1$. If the ρ_ℓ are a basis of semi-norms on V, set

$$\left\| \pi(X_k x^{2n}) v \right\|_{\rho_\ell} = M_{n,k,\ell} \,.$$

Let the α_n be such that $\sum \alpha_n M_{n,k,\ell} < \infty$ for all k, ℓ. Choose $0 < \varepsilon < 1/2$ and f, g smooth functions on $[-\varepsilon, \varepsilon]$ such that

$$\sum_0^p \alpha_n \frac{\delta^{(2n)}}{(2\pi)^{2n}} * f \to \delta + g \,.$$

Let $\mu = f(x)\,dx$, $\nu = g(x)\,dx$, identified through the exponential map with measures on G with support on $\exp(\mathbb{R}x) \cap \Omega$. Then

$$\pi(\mu) * \sum_0^p \alpha_n \pi(x^{2n}) v \to v + \pi(\nu) v$$

in V. The sequence

$$\sum_0^p \alpha_n \pi(x^{2n}) v$$

has as limit some η in V, and then $\pi(\mu)\eta = v + \pi(\nu)v$.

Let $\{x_j\}$ (for $1 \le j \le n$) be a basis of $\mathfrak{g}$. Applying this to $x = x_1$ gives us measures μ_0 and μ_1 with support on $\exp(\mathbb{R}x_1) \cap \Omega$ and α_0 in V^∞ such that $\pi(\mu_0)\alpha_0 = v + \pi(\mu_1)v$. If we let $\alpha_1 = v$ this gives us

$$v = \pi(\mu_0)\alpha_0 + \pi(\mu_1)\alpha_1 \,.$$

Applying it now to $x = x_2$ and each of the α_i gives us

$$v = \pi(\mu_{00})\alpha_{00} + \pi(\mu_{01})\alpha_{01} + \pi(\mu_{10})\alpha_{10} + \pi(\mu_{11})\alpha_{11} \,.$$

In this formula, each μ_{ij} is of the form $\nu_1 * \nu_2$ where ν_1 is a measure smooth on $\exp(\mathbb{R}x_1)$ and ν_2 is smooth on $\exp(\mathbb{R}x_2)$, hence itself smooth on the two-dimensional image of $\exp(\mathbb{R}x_1)\exp(\mathbb{R}x_2)$. Continuing $n-2$ more times we finally get

$$v = \sum_s \pi(\mu_s)\alpha_s$$

where s varies over the subsets of $\{1, \ldots, n\}$ expressed in bit notation and now the μ_s are smooth on Ω. ∎

7. References

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2. Jacques Dixmier and Paul Malliavin, 'Factorisations de fonctions et de vecteurs indéfiniment différentiables', *Bulletin des Sciences Mathématiques* **102** (1978), 305–330.