Math 100. Quiz 3. 2017-10-20 (Friday Q3-F-s) **Time 25min**

Section .... Instructor name ....

Your email ....

- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

1. Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) **(1pt)** Compute $f'(t)$ for $f(t) = \left(e^{2t} + t\right)^2$

Solution. Applying the chain rule:

$$f'(t) = 2(e^{2t} + t)(2 \cdot e^{2t} + 1) = 2(e^{2t} + t)(2e^{2t} + 1)$$

(b) **(1pt)** If $x^3y^2 + y = e^x$, compute $\dfrac{dy}{dx}$ at $(x, y) = (0, 1)$.

We compute $\frac{dy}{dx}$ by implicit differentiation. Differentiate both sides of the equation:

$$\frac{dy}{dx}\left(x^3y^2 + y\right) = \frac{dy}{dx}e^x$$

Then

$$3x^2y^2 + x^3(2y)\frac{dy}{dx} + \frac{dy}{dx} = e^x$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{e^x - 3x^2y^2}{2x^3y + 1} \tag{1}$$

Note that $(x, y) = (0, 1)$ satisfies the equation $x^3y^2 + y = e^x$. So we can evaluate the derivative (1) at this point:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)} = e^0 = 1.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Suppose $f(x)$ is a differentiable function such that $f(1) = 1$ and $f'(1) = 3$. Compute $g'(1)$ where

$$g(x) = f((f(x))^3)$$

Applying the chain rule:

$$g'(x) = f'(f(x)^3)(3f(x)^2)(f'(x))$$

Next we evaluate at $x = 1$:

$$g'(1) = f'(f(1)^3)(3f(1)^2)(f'(1)) = f'(1)(3 \cdot 1)(f'(1))$$

and hence $g'(1) = (3)^3 = 27$

(b) **(2pt)** Find all possible values for the constant C such that the tangent line to $y = \arcsin(Cx)$ at $x = 1$ is parallel to the line $2y - x = 7$.

Solution. First we find the derivative $\frac{dy}{dx}$ by taking the derivative of both sides of the equation:

$$\frac{dy}{dx} = \frac{dy}{dx}\arcsin(Cx) = \frac{C}{\sqrt{1-(Cx)^2}}$$

Evaluating at $x = 1$:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{C}{\sqrt{1-C^2}} \tag{2}$$

This is the slope of the tangent line at $x = 1$. The slope of the line $2y - x = 7$ is $\frac{1}{2}$. Recall that two lines are parallel if they have the same slope, so we are looking for the solutions to the equation

$$\frac{C}{\sqrt{1-C^2}} = \frac{1}{2} \tag{3}$$

Equation (3) implies $\frac{C^2}{1-C^2} = \frac{1}{4}$ **(it is not equivalent, and the previous posted solution is incorrect)**. Simplifying, this is $C^2 = 1/5$. Therefore $C = \pm\sqrt{1/5}$. Between these two values the one that satisfies (3) is $\underline{C = \sqrt{1/5}}$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Consider the following equation

$$\frac{x}{y-1} = x^{y+1}$$

Compute $\dfrac{dy}{dx}$ at the point $(x, y) = (1, 2)$.

Solution. In order to compute y' we will use logarithmic differentiation.

$$\log\left(\frac{x}{y-1}\right) = \log\left(x^{y+1}\right)$$

Simplifying

$$\log(x) - \log(y-1) = (y+1)\log(x)$$

Next we differentiate

$$\frac{d}{dx}\left(\log(x) - \log(y-1)\right) = \frac{d}{dx}(y+1)\log(x)$$

Then

$$\frac{1}{x} - \frac{1}{y-1}\frac{dy}{dx} = \frac{y+1}{x} + \log(x)\frac{dy}{dx} \tag{4}$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \left(\frac{1}{x} - \frac{y+1}{x}\right) \Big/ \left(\log(x) + \frac{1}{y-1}\right)$$

Note that $(x, y) = (1, 2)$ satisfies $\frac{x}{y-1} = x^{y+1}$, so we evaluate the derivative at this point:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = (1-3) \Big/ (\log(1) + 1) = \frac{1-3}{0+1} = -2$$

Math 100. Quiz 3. Solutions 2017-10-20 (Friday) **Time 25min**

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- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

1. Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) **(1pt)** Compute $f'(x)$ for $f(x) = \sqrt{1 + \cos(2\pi x)}$
Solution. Applying the chain rule:

$$f'(x) = \frac{-2\pi \sin(2\pi x)}{2\sqrt{1 + \cos(2\pi x)}} = \frac{-\pi \sin(2\pi x)}{\sqrt{1 + \cos(2\pi x)}}$$

(b) **(1pt)** If $xy + y^2x + 1 = x^2$, compute $\dfrac{dy}{dx}$ at $(x, y) = (1, 0)$.

Solution. We compute $\frac{dy}{dx}$ by implicit differentiation. Differentiate both sides of the equation:

$$\frac{d}{dx}\left(xy + y^2x + 1\right) = \frac{d}{dx}x^2$$

Then

$$y + x\frac{dy}{dx} + y^2 + x(2y)\frac{dy}{dx} = 2x$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x - y^2 - y}{x + 2xy} \tag{1}$$

Note that $(x, y) = (1, 0)$ satisfies the equation $xy + y^2x + 1 = x^2$. So we can evaluate the derivative (1) at this point:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,0)} = 2.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Suppose $f(x)$ is a differentiable function such that $f(1) = 1$ and $f'(1) = 2$. Compute $g'(1)$ where

$$g(x) = f(f(x^3))$$

Solution. Applying the chain rule:

$$g'(x) = f'(f(x^3))f'(x^3)(3x^2)$$

Next we evaluate at $x = 1$:

$$g'(1) = f'(f(1^3))f'(1^3)(3 \cdot 1^2) = f'(f(1))f'(1)(3)$$

and hence $g'(x) = f'(1)f'(1)(3) = 12$

(b) **(2pt)** Find all possible values for the constant C such that the tangent line to $y = C\arctan(Cx)$ at $x = 1$ is parallel to the line $3y - x = 1$.

Solution. First we find the derivative $\frac{dy}{dx}$ by taking the derivative of both sides of the equation:

$$\frac{dy}{dx} = \frac{dy}{dx}C\arctan(Cx) = C\frac{dy}{dx}\arctan(Cx) = \frac{C^2}{1+(Cx)^2}$$

Evaluating at $x = 1$:

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{C^2}{1+C^2} \tag{2}$$

This is the slope of the tangent line at $x = 1$. The slope of the line $3y - x = 1$ is $\frac{1}{3}$. Recall that two lines are parallel if they have the same slope, so we are looking for the solutions to the equation

$$\frac{C^2}{1+C^2} = \frac{1}{3} \tag{3}$$

Equation (3) is equivalent to $3C^2 = 1 + C^2$. Simplifying, this is $C^2 = 1/2$. Therefore $C = \pm\sqrt{1/2}$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Consider the following equation

$$4xy = (x^2+1)^{y+1}$$

Compute $\dfrac{dy}{dx}$ at the point $(x, y) = (1, 1)$.

Solution. In order to compute y' we will use logarithmic differentiation.

$$\log(4xy) = \log((x^2+1)^{y+1})$$

Simplifying

$$\log(4) + \log(x) + \log(y) = (y+1)\log(x^2+1)$$

Next we differentiate

$$\frac{d}{dx}\left(\log(4) + \log(x) + \log(y)\right) = \frac{d}{dx}(y+1)\log(x^2+1)$$

Then

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = (y+1)\frac{2x}{x^2+1} + \log(x^2+1)\frac{dy}{dx} \tag{4}$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \left(\frac{2x(y+1)}{x^2+1} - \frac{1}{x}\right) \Big/ \left(\frac{1}{y} - \log(x^2+1)\right)$$

Note that $(x, y) = (1, 1)$ satisfies $4xy = (x^2+1)^{y+1}$, so we evaluate the derivative at this point:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \left(\frac{2\cdot 2}{2} - 1\right) \Big/ (1 - \log 2)) = \frac{1}{1-\log 2}$$