Quiz 2

2017-10-05

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** For $f(x) = (1 + x^2)\sqrt{x}$, compute $f'(1)$.
Solution. The derivative of $f(x)$ is

$$f'(x) = (2x)\sqrt{x} + (1 + x^2)\left(\frac{1}{2\sqrt{x}}\right)$$

then $f'(1) = 2 + (2)(1/2) = 2 + 1 = 3$.

(b) **(1pt)** There is a car on a highway, whose location at time t is given by $y(t) = 80t + 30\cos t$. Find its instantaneous speed at $t = \pi/2$. (Ignore the units.)
Solution. The derivative of $y(t)$ is $y'(t) = 80 - 30\sin t$. The instantaneous speed at $t = \pi/2$ is

$$y'(\pi) = 80 - 30\sin(\pi/2) = 80 - 30 = 50.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Find the equation of the tangent line to the graph of $y = \sin x + e^x$ at $x = 0$.

Solution. The derivative of $y = \sin x + e^x$ is $y' = \cos x + e^x$, and so the slope of the tangent line at $x = 0$ is $\cos 0 + e^0 = 2$. The equation of the tangent line is $y = (\sin 0 + e^0) + 2x$, i.e., $y = 1 + 2x$.

(b) **(2pt)** Show that there is a real number x such that $x^2 - 1 = \tan(x)$.

Solution. We let $f(x) = x^2 - 1 - \tan(x)$. This is a continuous function on $[-\pi/4, 0]$. We compute

$$f(-\pi/4) = \pi^2/16 - 1 - (-1) = \pi^2/16 > 0$$

and

$$f(0) = 0 - 1 - 0 = -1 < 0.$$

Therefore, by the Intermediate Value Theorem, there exists $x \in (-\pi/4, 0)$ such that $f(x) = 0$, as desired.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Use the definition of the derivative to find a and b such that the following function

$$f(x) = \begin{cases} x^5 + ax + b & \text{if} \quad x \leq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x > 0 \end{cases}$$

is differentiable at $x = 0$. You must justify your answer.

Solution. If $f(x)$ is differentiable at $x = 0$, then $f(x)$ is continuous at $x = 0$ and so,

$$\lim_{x \to 0} f(x) = f(0).$$

We compute $f(0) = b$, while $\lim_{x\to 0} f(x)$ is computed using both lateral limits:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^5 + ax + b = b,$$

while $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin(1/x) = 0$ by Squeeze Theorem. Indeed, $-1 \leq \sin(1/x) \leq 1$ and so, $-x^2 \leq x^2 \sin(1/x) \leq x^2$ and so, as $x \to 0^+$, we get that $x^2 \sin(1/x) \to 0$, as claimed. So, $f(x)$ is continuous at $x = 0$ if $b = 0$. In particular, this means that $f(0) = b = 0$.

Now, in order for $f(x)$ be differentiable at $x = 0$ we need the following limit to exist:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} \frac{f(x)}{x}.$$

Again we need to compute both lateral limits. We get

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0^+} x \sin(1/x) = 0,$$

by Squeeze Theorem, again. Indeed, $-x \leq x \sin(1/x) \leq x$ for $x \to 0^+$ and so, the Squeeze Theorem yields that $\lim_{x\to 0^+} x \sin(1/x) = 0$ since $\lim_{x\to 0^+} x = \lim_{x\to 0^+} -x = 0$.

We also compute

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{x^5 + ax}{x} = \lim_{x \to 0^-} x^4 + a = a.$$

Therefore, $f(x)$ is differentiable at $x = 0$ if $a = 0$.

Q2-t-s-solution

Math 100. Quiz 2. 2017-10-05 Thursday. **Time 25min.**

Section .... Instructor name ....

Your email ....

- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{150}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

1. Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) **(1pt)** For $f(x) = \dfrac{1}{\sqrt{x}+1}$, compute $f'(1)$.

Solution. The derivative of $f(x)$ is

$$f'(x) = -\frac{1}{2\sqrt{x}}\frac{1}{(\sqrt{x}+1)^2}$$

then $f'(1) = -(\frac{1}{2})(\frac{1}{2^2}) = -\frac{1}{8}$.

(b) **(1pt)** There is a cyclist on 10th Avenue, whose location at time t is given by $y(t) = 15t - 5\sin t$. Find its instantaneous speed at $t = \pi$. (Ignore the units.)

Solution. The derivative of $y(t)$ is $y'(t) = 15 - 5\cos t$. The instantaneous speed at $t = \pi$ is

$$y'(\pi) = 15 - 5\cos(\pi) = 15 + 5 = 20.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Find the equation of the tangent line to the graph of $y = 2e^x + \cos x$ at $x = 0$.

Solution. The derivative of $y = 2e^x + \cos x$ is $y' = 2e^x - \sin x$, and so the slope of the tangent line at $x = 0$ is $y'(0) = 2$. The equation of the tangent line is $y = (2e^0 + cos(0)) + 2(x - 0)$, i.e. $y = 3 + 2x$.

(b) **(2pt)** Show that there is a real number x satisfying the equation

$$2x^2 = \tan(x) + 1$$

Solution. We let $f(x) = 2x^2 - \tan(x) - 1$. This is a continuous function on $[-\pi/4, 0]$. We compute

$$f(-\pi/4) = 2\pi^2/16 - 1 - (-1) = \pi^2/8 > 0$$

and

$$f(0) = 0 - 1 - 0 = -1 < 0.$$

Therefore, by the Intermediate Value Theorem, there exists $x \in (-\pi/4, 0)$ such that $f(x) = 0$, as desired.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Use the definition of the derivative to find a and b such that the following function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + a & \text{if } x < 0 \\ 3x^2 + (2+b)x & \text{if } x \geq 0. \end{cases}$$

is differentiable at $x = 0$. You must justify your answer.

Solution. If $f(x)$ is differentiable at $x = 0$, then $f(x)$ is continuous at $x = 0$ and so,

$$\lim_{x\to 0} f(x) = f(0) = 0. \tag{1}$$

We compute $\lim_{x\to 0} f(x)$ using both lateral limits. The right-hand limit is $\lim_{x\to 0^+} 3x^2 + (2+b)x = 0$, as desired. The left-hand limit exists by the Squeeze Theorem: Indeed, since $-1 \leq \sin(1/x) \leq 1$ and so, $-x^2 \leq x^2 \sin(1/x) \leq x^2$, we have $\lim_{x\to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$. Therefore

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin\left(\frac{1}{x}\right) + a = a$$

In order to satisfy equality (1) above we need $a = 0$, and hence $f(x)$ is continuous at $x = 0$.

Now, in order for $f(x)$ be differentiable at $x = 0$ we need the following limit to exist:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} \frac{f(x)}{x}.$$

Again we need to compute both lateral limits. We get

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0^-} x \sin(1/x) = 0,$$

by Squeeze Theorem again. Indeed, $-x \leq x \sin(1/x) \leq x$ for $x \to 0^+$ and so, the Squeeze Theorem yields that $\lim_{x\to 0^-} x \sin(1/x) = 0$ since $\lim_{x\to 0^-} x = \lim_{x\to 0^-} -x = 0$.

We also compute

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{3x^2 + (2+b)x}{x} = \lim_{x\to 0^+} 3x + (2+b) = 2+b.$$

Since $f(x)$ is differentaible at $x = 0$ if

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{f(x)}{x},$$

we need $2 + b = 0$ and therefore $b = -2$.