Q2-F-p-solution

Quiz 2

2017-10-06

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Find a positive integer n such that $x^3 - 3x = 4$ for some x in the interval $[n, n+1]$.

Solution. Let $f(x) = x^3 - 3x - 4$. We have $f(1) = -6$, $f(2) = 8 - 6 - 4 = -2$, $f(3) = 27 - 9 - 4 > 0$. By Intermediate Value Theorem, $f(x) = 0$ for some $x \in [2, 3]$. Take $n = 2$.

(b) **(1pt)** Compute the derivative of $\dfrac{3x-2}{2x+5}$. Simplify your answer.

Solution. We use the quotient rule,

$$\begin{aligned} \left(\frac{3x-2}{2x+5}\right)' &= \frac{3(2x+5) - (3x-2)2}{(2x+5)^2} \\ &= \frac{19}{(2x+5)^2} \end{aligned}$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Use the definition of continuity to find all values a and b such that

$$f(x) = \begin{cases} \dfrac{x^2 + 2x + a}{x} & x > 0 \\ b - 3x & x \leq 0 \end{cases}$$

is continuous everywhere.

Solution. We need

$$\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x).$$

We have $\lim_{x \to 0^-} f(x) = f(0) = b$. Hence it suffices to have $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x^2+2x+a}{x} = b$.

For the fraction $\frac{x^2+2x+a}{x}$, the denominator converges to 0 as $x \to 0$. Thus we need the numerator also to converge to 0, i.e., $\lim_{x \to 0^+} x^2 + 2x + a = 0$. So **a=0**.

Then we need

$$b = \lim_{x \to 0^+} \frac{x^2 + 2x}{x} = \lim_{x \to 0^+} \frac{x(x+2)}{x} = \lim_{x \to 0^+} x + 2 = 2$$

That is **b=2**.

(b) **(2pt)** Find the x-coordinates of the points on the graph of $y = x^3 - 5x$ where the tangent line is parallel to $70x + 1$.

Solution. The slope of the tangent is $f'(x) = 3x^2 - 5$; so we are asked for this slope to equal 70 (since this means that two lines are parallel— they must have the same slope). We solve and obtain $3x^2 - 5 = 70$, i.e., $x^2 = 25$ and therefore $x = -5$ or $x = 5$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find all positive real numbers a with the property that the function

$$f(x) = \begin{cases} x^a\left(\cos(1/x) - 2\right) & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

is differentiable at $x = 0$. **Justify your answer using the definition of the derivative.**

Solution. In order for $f(x)$ be differentiable at $x = 0$ we need the following limit to exist:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}.$$

Again we need to compute both lateral limits. We get

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0}{x} = 0,$$

where we used the fact that $f(0) = 0$. Hence, we need

$$0 = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} x^{a-1}\left(\cos(1/x) - 2\right).$$

If $a > 1$, then since $-1 - 2 \leq \cos(1/x) - 2 \leq 1 - 2$, we have that when $x \to 0^+$:

$$-3x^{a-1} \leq x^{a-1}\left(\cos(1/x) - 2\right) \leq -x^{a-1}.$$

Since $a > 1$, both $-3x^{a-1}$ and $-x^{a-1}$ converge to 0 as $x \to 0^+$. Hence, by the Squeeze Theorem,

$$\lim_{x \to 0^+} x^{a-1}\left(\cos(1/x) - 2\right) = 0.$$

So when $a > 1$, f is differentiable at $x = 0$.

If $a = 1$, then $\lim_{x \to 0^+}\left(\cos(1/x) - 2\right)$ is undefined because of the oscillating function $\cos(1/x)$. So when $a = 1$, f is NOT differentiable at $x = 0$.

If $a < 1$, then $\lim_{x \to 0^+} x^{a-1}\left(\cos(1/x) - 2\right) = -\infty$ because

$$\lim_{x \to 0^+} x^{a-1} = \lim_{x \to 0^+} \frac{1}{x^{1-a}} = \infty.$$

So when $a < 1$, f is NOT differentiable at $x = 0$.

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For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Find a positive integer n such that $x^3 - 1 = 5x$ for some x in the interval $[n, n+1]$.

Solution. Let $f(x) = x^3 - 5x - 1$. We have $f(1) = 1 - 5 - 1 = -5$, $f(2) = 8 - 10 - 1 = -3$, $f(3) = 27 - 15 - 1 = 11$. By Intermediate Value Theorem, $f(x) = 0$ for some $x \in [2, 3]$. Take $n = 2$.

(b) **(1pt)** Compute the derivative of $\frac{5x-4}{3x+2}$. Simplify your answer.

Solution. We use the quotient rule,

$$\begin{aligned}\left(\frac{5x-4}{3x+2}\right)' &= \frac{5(3x+2) - (5x-4)3}{(3x+2)^2} \\ &= \frac{22}{(3x+2)^2}\end{aligned}$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Use the definition of continuity to find all values a and b such that

$$f(x) = \begin{cases} \dfrac{x^2 - a}{x - 1} & x > 1 \\ b - 3x + x^2 & x \leq 1 \end{cases}$$

is continuous everywhere.

Solution. We need

$$\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x).$$

We have $\lim_{x \to 1^-} f(x) = f(1) = b - 3 + 1 = b - 2$. Hence it suffices to have $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2 - a}{x - 1} = b - 2$.

For the fraction $\frac{x^2-a}{x-1}$, the denominator converges to 0 as $x \to 1$. Thus we need the numerator also to converge to 0, i.e., $\lim_{x \to 1^+} x^2 - a = 0$. So $1 - a = 0$, i.e., **a=1**.

Then we need

$$b - 2 = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^+} x + 1 = 1 + 1 = 2$$

That is **b=4**.

(b) **(2pt)** Find the x-coordinates of the points on the graph of $y = x^3 + 3$ where the tangent line is parallel to $48x + 48$.

Solution. The slope of the tangent is $f'(x) = 3x^2$; so we are asked for this slope to equal 48 (since this means that two lines are parallel—they must have the same slope). We solve and obtain $3x^2 = 48$, i.e., $x^2 = 16$ and therefore $x = -4$ or $x = 4$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find all positive real numbers a with the property that the function

$$f(x) = \begin{cases} x^a(\sin(1/x)+2) & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \le 0 \end{cases}$$

is differentiable at $x = 0$. **Justify your answer using the definition of the derivative.**

Solution. In order for $f(x)$ be differentiable at $x = 0$ we need the following limit to exist:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

Again we need to compute both lateral limits. We get

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{0}{x} = 0,$$

where we used the fact that $f(0) = 0$. Hence, we need

$$0 = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} x^{a-1}(\sin(1/x)+2).$$

If $a > 1$, then since $-1 + 2 \le \sin(1/x) + 2 \le 1 + 2$, we have that when $x \to 0^+$:

$$x^{a-1} \le x^{a-1}(\sin(1/x)+2) \le 3x^{a-1}.$$

Since $a > 1$, both x^{a-1} and $3x^{a-1}$ converge to 0 as $x \to 0^+$. Hence, by the Squeeze Theorem,

$$\lim_{x\to 0^+} x^{a-1}(\sin(1/x)+2) = 0.$$

So when $a > 1$, f is differentiable at $x = 0$.

If $a = 1$, then $\lim_{x\to 0^+}(\sin(1/x)+2)$ is undefined because of the oscillating function $\sin(1/x)$. So when $a = 1$, f is NOT differentiable at $x = 0$.

If $a < 1$, then $\lim_{x\to 0^+} x^{a-1}(\sin(1/x)+2) = \infty$ because

$$\lim_{x\to 0^+} x^{a-1} = \lim_{x\to 0^+} \frac{1}{x^{1-a}} = \infty.$$

So when $a < 1$, f is NOT differentiable at $x = 0$.