Review 1

- This is an optional problemset.
- If you want feedback on your work, please turn it in on Wednesday Feb. 3rd in class. The solution will be posted soon thereafter.
- If you decide to turn it in, your work should be clean and organized : if it is not easy to read, it won't be read.

Problem 1. Parameterize the curve $y = \frac{2}{3}x^{3/2}$ with respect to arc length in the direction of increasing x.

Problem 2. For a curve $\mathbf{r}(t)$, find an expression for $\frac{d\mathbf{N}}{dt} \cdot \mathbf{T}$ in terms of $|\mathbf{r}'(t)|$ and curvature κ. Here, $\mathbf{N}$ is the principal normal vector, and $\mathbf{T}$ is the unit tangent vector (hint : differentiate $\mathbf{N} \cdot \mathbf{T}$).

Problem 3. Calculate the arc length of the curve parameterized by

$$x(t) = \cos^3(t)\,, \quad y(t) = \sin^3(t); \qquad 0 \le t \le \pi\,.$$

Problem 4. Let C be the intersection of the cylinder $x^2 + y^2 = 9$ and the plane $x + z = 0$. We are looking for the points where the curvature is maximal.

(1) Can you guess before doing any calculation how many points you are going to find ? Explain.

(2) Give a parametrization of C of the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

(3) Find the points where the curvature is maximal.

(4) Let

$$\mathbf{u} := \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{k}, \quad \mathbf{v} := \mathbf{j}, \quad \mathbf{w} := \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{k}.$$

Write $\mathbf{r}(t)$ in the form

$$a(t)\mathbf{u} + b(t)\mathbf{v} + c(t)\mathbf{w}$$

where $a(t)$, $b(t)$ and $c(t)$ are functions you have to determine.

(5) Draw the curve C in the system of coordinates $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Problem 5. Consider the ellipse $x^2 + 2y^2 = 18$. What is the equation of the osculating circle at point $(0, 3)$? Draw the curve and this osculating circle. *To compute the normal vector, you can either use the formula learned in class (which might be complicated) or use the geometric description of this vector and figure out its coordinates.*

Problem 6. (1) Calculate the work done by the vector field $\mathbf{F}(x, y) = (xy, 3y^2)$ on a particle moving

(a) along the path $\mathbf{r}(t) = (11t^4, t^3)$ from $t = 0$ to $t = 1$;

(b) along the line segment from $(0, 0)$ to $(11, 1)$.

(2) Is the vector field $\mathbf{F}$ conservative ?

Problem 7. Let $a \in \mathbb{R}$. Consider the vector field

$$\mathbf{F}(x, y, z) = (\cos(x) + (a - 3)y, 2 + \cos(y), e^z).$$

(1) Find the value of a for which the vector field is conservative and give f such that $\mathbf{F} = \nabla f$.

(2) For the value of a you found in (1), compute the line integral of $\mathbf{F}$

  (a) along the line segment from $(\pi/2, \pi/2, 0)$ to $(\pi, \pi, 2)$;

  (b) along the curve $\mathbf{r}(t) = (\cos(t) + 1, \sin(t), 3\cos(t) + 4\sin(t))$ with $0 \leq t \leq 2\pi$;

  (c) along the spiral curve $\mathbf{r}(t) = (t, \cos(t), \sin(t))$ with $0 \leq t \leq 3\pi$.

(3) Compute the integral

$$\int_C (\cos(x) + 7y)dx + (2 + \cos(y))dy + e^z dz$$

where C is the curve (b) above with as little calculation as possible.