Suggested problems or various things to think about

Problem 1. Consider the symmetric group $\mathcal{S}_3$ and let k be an arbitrary field of characteristic different from 2.

(1) What are the 1-dimensional representation of $\mathcal{S}_3$? (Show that there is only the trivial representation and the character corresponding to the signature that we will call the sign character).

(2) Consider the natural 3-dimensional representation V_3 of $\mathcal{S}_3$ over k. What is the multiplicity of the trivial character of $\mathcal{S}_3$ in V_3 ? Does V_3 contain the sign character as a subrepresentation ? Show that V_3 contains a subrepresentation of dimension 2.

(a) If k has characteristic different from 3, show that V_3 is semisimple and give its decomposition into irreducibles.

(b) If k has characteristic 3, give a filtration of V_3 as a representation of $\mathcal{S}_3$ of the form
$$0 \subset V_1 \subset V_2 \subset V_3$$
where V_i has dimension i. Describe the quotient representations V_{i+1}/V_i. Show that V_2 is indecomposable and that V_3 is indecomposable

What happens over a field of characteristic 2 ?

Problem 2. Let G be a finite group. Consider its trivial representation k_{triv} : it is one dimensional and it corresponds to the morphism of groups $G \to k^\times$, $g \mapsto 1$. It is also often denoted by 1_G. We suppose that the cardinality of G is invertible in the field k. Consider the element
$$e_1 := \frac{1}{|G|} \sum_{g \in G} g \in k[G].$$

(1) Show that it is a central idempotent of $k[G]$ that is to say that $e_1^2 = e_1$ and that e_1 commutes with any element of $k[G]$. Deduce that the trivial representation is a direct summand of the regular representation of G over k.

(2) Consider an exact sequence of k-representations of G :
$$0 \longrightarrow V \longrightarrow W \longrightarrow k_{triv} \longrightarrow 0$$
This means that there are morphisms of representations $\iota : V \to W$ and $p : W \to k_{triv}$ such that ι is injective, p is surjective, and $\mathrm{Im}(\iota) = \mathrm{Ker}(p)$. Show that the exact sequence splits that is to say that there is $s : k_{triv} \to W$ a morphism of representations such that $\iota \circ s = id_{k_{triv}}$. *(This means that k_{triv} is a projective representation of G).* Check that it implies that $W \simeq V \oplus k_{triv}$ as representations of G.

(3) Show that if k_{triv} is a quotient of a k-representation W, then W contains a copy of k_{triv} as a direct summand. *This also means that k_{triv} is a projective representation of G.*

To be continued.

Problem 3. Let U be the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{F}_p)$. What is the cardinality of U ? Consider the natural representation of U on the vector space $\mathbb{F}_p^n$. Is is decomposable ? Is the trivial representation of U over $\mathbb{F}_p$ projective ?

Problem 4. Let G be a group and k a field.

(1) Let $[G, G]$ be the subgroup of commutators of G i.e the subgroup generated by all $[g, g'] := gg'g^-1g'^{-1}$ for $g, g' \in G$. What is $[G, G]$ if G is abelian ? Show that $[G, G]$ is a normal subgroup of G. Show that any morphism of groups $G \to H$ where H is abelian factors through $G/[G, G] \to H$. Note that $[G, G]$ is the smallest normal subgroup of G such that the quotient group of G by this subgroup is abelian. We denote $G/[G, G]$ by G_{ab} and call it the abelianization of G. For example, when $n \geq 3$, the commutator subgroup of $\mathcal{S}_n$ is $\mathcal{A}_n$: what is the abelianization of $\mathcal{S}_n$?

(2) We are interested in the k-representations V of G such that there is an exact sequence of representations :

$$0 \to k_{triv} \to V \to k_{triv} \to 0.$$

These are called extension of k_{triv} by itself. Show that to such a representation V, one can associate naturally a morphism of groups $\varphi_V : G \to k$. What does it mean for V if $\varphi_V = 0$?

(3) Suppose that G is finite.
   (a) Suppose that the characteristic of k does not divide its cardinality. Describe the morphisms of groups $G \to k$. How many non isomorphic extensions of k_{triv} by itself are there ?
   (b) Suppose that U is the subgroup of upper triangular unipotent matrices in $\mathrm{GL}_2(\mathbb{F}_p)$ and that k has characteristic p. Are there non semisimple extensions of k_{triv} by itself ? Can you describe all of them ?

The set of morphisms of groups $\mathrm{Hom}(G, k)$ *is the first cohomology group* $\mathrm{H}^1(G, k)$

Problem 5. Let G be a finite group and H a subgroup of G. Let k be a field. Consider the induced representation

$$\mathrm{ind}_H^G(\mathrm{triv}_H) \cong k[G/H]$$

where triv_H denotes the trivial representation of H. A basis for $k[G/H]$ is given by the characteristic functions 1_{gH} where g ranges over a system of representatives of the left cosets G/H.

(1) Show that the algebra of endomorphisms $\mathcal{H} := \mathrm{End}_G(k[G/H])$ identifies naturally with the algebra $k[H\backslash G/H]$ of all functions on $H\backslash G/H$ with values in k (which can also be seen as functions on G that are constant of the double cosets modulo H). The product of φ and ψ in $k[H\backslash G/H]$ is given by

$$\varphi \star \psi(x) = \sum_{g \in G/H} \psi(g)\varphi(g^{-1}x).$$

(2) Show that there is a natural functor

$$\mathcal{F} : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Mod}(\mathcal{H}), \quad V \longmapsto V^H$$

where $\mathrm{Mod}(\mathcal{H})$ is the category of right $\mathcal{H}$-modules.

(3) Give a condition under which you can prove that $\mathcal{F}$ is exact. Show that it is not exact in general.

(4) Show that the functor $- \otimes_{\mathcal{H}} k[G/H]$ is left adjoint to $\mathcal{F}$.

(5) Let $\mathrm{Rep}_k^H(G)$ denote the full subcategory of $\mathrm{Rep}_k(G)$ of the representations generated by their H-fixed vectors. Show that the restriction of $\mathcal{F}$ to $\mathrm{Rep}_k^H(G)$ is faithful.

(6) Suppose that $G = \mathrm{GL}_2(\mathbb{F}_q)$ and $H = B$ is the upper Borel subgroup.
   (a) Show that $\mathcal{H}$ is two dimensional as a vector space with basis 1_B and 1_{BsB}.
   (b) Compute $1_{BsB} \star 1_{BsB}$.
   (c) Show that $\mathcal{H}$ is semisimple and describe its simple modules.

(d) Show that the functor above matches up irreducible representations in $\mathrm{Rep}_k^B(G)$ and simple modules of $\mathcal{H}$.

Problem 6. Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ and B be the upper Borel subgroup with Levi decomposition $B = TU$. Let $\overline{U}$ denote the lower unipotent subgroup.

(1) Consider the symmetric group $\mathfrak{S}_n$ and let S denote the set of all transpositions of the form $s_i := (i, i+1)$ for $i = 1, \ldots, n-1$. We say that $(\mathfrak{S}_n, S)$ is a Coxeter system. In particular, it means that any element of $\mathfrak{S}_n$ can be written as a product of elements in S. The length of a word $s_{i_1} \ldots s_{i_m}$ for $m \geq 0$ and $1 \leq i_1, \ldots, i_n \leq n-1$, is the integer m. For $w \in \mathfrak{S}_n$, if $w = s_{i_1} \ldots s_{i_m}$ we say that it is a decomposition of length m of w. We denote by $\ell(w)$ the minimal length of a decomposition of w. A decomposition of w of length $\ell(w)$ is called reduced. The length of $1 \in \mathfrak{S}_n$ is by definition zero.

(a) For $w, w' \in \mathfrak{S}_n$ show that $\ell(ww') \leq \ell(w) + \ell(w')$. Give examples where the inequality is strict (resp. where it is an equality).

(b) If $w = s_{i_1} \ldots s_{i_{\ell(w)}}$ is a reduced decomposition for w and $1 \leq j < \ell(w)$, show that $s_{i_1} \ldots s_{i_j}$ has length j and $s_{j_{i+1}} \ldots s_{i_{\ell(w)}}$ has length $\ell(w) - j$.

(c) For $s \in S$ and $w \in \mathfrak{S}_n$, show that $\ell(w) - 1 \leq \ell(ws) \leq \ell(w) + 1$.

(2) We consider the set of pairs of integers

$$\Phi := \{(i,j),\ i \neq j,\ 1 \leq i, j \leq n\}.$$

It is the union of Φ^+ and Φ^- where

$$\Phi^+ = \{(i,j) \in \Phi,\ i < j\} \text{ and } \Phi^- = \{(i,j) \in \Phi,\ i > j\}.$$

The set Φ is the set of roots of G and Φ^+ (resp. Φ^-) is the set of the positive (resp. negative) roots.

To $\alpha = (i,j) \in \Phi$ we attach the subgroup $U_\alpha := 1 + \mathbb{F}_q e_{i,j}$ of G where $e_{i,j}$ is the $n \times n$ matrix with zero coefficients except for the coefficient (i,j) which is equal to 1.

(a) What is the cardinality of Φ, of Φ^+?

(b) Show that to $s \in S$, one can attach naturally an element α_s in Φ^+. We denote by Π the set of all $\{\alpha_s, s \in S\}$. Check that every element of Φ^+ is a sum of distinct elements of Π. Is any sum of distinct elements in Π an element in Φ^+?

(c) Check that U is the product of all U_α for $\alpha \in \Phi^+$ (this is the decomposition of U into root subgroups).

(d) Check that there is a natural action of $\mathfrak{S}_n$ on Φ which is compatible with the action by conjugation of $\mathfrak{S}_n$ on the subgroups U_α for $\alpha \in \Phi$.

(e) Let $s \in S$. What is the set $\{\alpha \in \Phi^+,\ s.\alpha \in \Phi^-\}$?

(f) Check on some examples (e.g when $n = 2$ and $n = 3$) that for $w \in \mathfrak{S}_n$, we have

$$\ell(w) = |\{\alpha \in \Phi^+,\ w.\alpha \in \Phi^-\}| = |\{\alpha \in \Phi^+,\ wU_\alpha w^{-1} \subset \overline{U}\}|.$$

(g) Show that for any $s \in S$ and $w \in \mathfrak{S}_n$, we have $\ell(ws) = \begin{cases} \ell(w) + 1 & \text{if } w.\alpha_s \in \Phi^+ \\ \ell(w) - 1 & \text{if } w.\alpha_s \in \Phi^- \end{cases}$

(3) Show that for $s \in S$ we have $BsB = \bigcup_{u \in U_{\alpha_s}} usB$.

(4) Show that for any $s \in S$ and $w \in \mathfrak{S}_n$ such that $\ell(ws) = \ell(w) + \ell(s)$ we have

$$BwBsB = BwsB.$$

(5) Show that for any $w, w' \in \mathfrak{S}_n$ such that $\ell(ww') = \ell(w) + \ell(w')$ we have
$$BwBw'B = Bww'B.$$

(6) What is $BsBsB$?

(7) Let k be an arbitrary field. Show that the Hecke algebra $\mathcal{H}$ of G with respect to B has k-basis the set of all characteristic functions $\tau_w := 1_{BwB}$ for $w \in \mathfrak{S}_n$ subject to the relations
$$\tau_w \star \tau_{w'} = \tau_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w'),$$
$$\tau_s^2 = \tau_s \star \tau_s = (q-1)\tau_s + q \quad \text{for any } s \in S.$$

(a) What are the one dimensional modules of $\mathcal{H}$?

(b) Suppose that k has characteristic p, show that any simple $\mathcal{H}$ module is one dimensional. Is $\mathcal{H}$ semisimple ? Justify your answer.

(c) Suppose that $n = 3$. Give the decomposition of $\mathcal{H}$ into PIMs and identify the projective covers of the simple modules.

Problem 7. Let A be an abelian group. Decompose the unit of $\mathbb{C}[A]$ as a sum of primitive orthogonal idempotents.

Problem 8. Let A be a k-algebra. A left A-module M is flat if the functor $- \otimes_A M$ from right A-modules to k-vectorspaces is exact.

(1) Show that if M is flat then for any right ideal $\mathcal{I}$ of A, the map $\mathcal{I} \otimes_A M \to M$ is injective. We admit that this condition is sufficient for the flatness of M (but you can think/lookup the proof).

(2) Show that any direct summand of a flat module is flat.

(3) Show that a projective module is flat.

Problem 9. Let G be a finite group and k a field. Show that $k[G]$ is an injective $k[G]$-module by proving that $\mathrm{Hom}_G(-, k[G])$ is exact. (Note that we can also deduce this from the fact that $k[G]$ is a symmetric algebra...)

Problem 10. Let A, B be unitary rings and $B \to A$ a morphism of rings making A a left and right A-module. We consider the induction functor $A \otimes_B -$ from left B-modules to left A-modules.

(1) Show that $A \otimes_B -$ preserves the projectivity of the modules. Does it take a projective resolution of the B-module M to a projective resolution of the A-module $A \otimes_B M$?

(2) Suppose $A = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}$.

(a) Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$.

(b) Describe a non split extension of $\mathbb{Z}/2\mathbb{Z}$ by itself as an abelian group.

(c) Compute $\mathrm{Ext}^i_A(A \otimes_B \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for $i \geq 0$. What do you notice when you compare with (a) ?

(3) Suppose that A is a free right B-module. Show that $A \otimes_B -$ is exact. What can you say about $\mathrm{Ext}^i_A(A \otimes_B -, -)$?

(4) Same question when A is a projective right B-module.

(5) Same question when A is a flat right B-module.

(6) Suppose that G is a group and H a subgroup of G. Let k be a field. What can you say about $\mathrm{Ext}^i_{k[G]}(k[G] \otimes_{k[H]} -, -)$?

Problem 11. Let G be a finite group and J a subgroup of G. Let k be a field. Let $i \geq 0$.

(1) Show that for a $k[J]$-module N and a $k[G]$-module M, we have
$$\mathrm{Ext}^i_{k[G]}(M, \mathrm{ind}_J^G(N)) = \mathrm{Ext}^i_{k[J]}(M|_J, N).$$

(2) Show that
$$H^i(J, k) = \mathrm{Ext}^i_{k[G]}(k, \mathrm{ind}_H^G(k)).$$

Problem 12. Let q be a power of a prime number p and $G = \mathrm{GL}_n(\mathbb{F}_q)$. Let U be the subgroup of all upper unipotent matrices and B the Borel subgroup containing U with Levi decomposition $B = TU$. Let k be a field and $\chi : T \to k^\times$ a morphism of groups. We may consider it a morphism of groups $B \to k^\times$ trivial on U. For any $w \in \mathfrak{S}_n$ let $U_w := U \cap^w Uw^{-1}$.

(1) Using Mackey decomposition, show that
$$\mathrm{ind}_B^G(\chi)|_U \cong \bigoplus_{w \in \mathfrak{S}_n} \mathrm{ind}_{U_w}^U(k).$$

(2) Show that
$$H^1(U, \mathrm{ind}_B^G(\chi)) = \bigoplus_{w \in \mathfrak{S}_n} \mathrm{Hom}(U_w, k).$$

(3) What is $H^1(U, \mathrm{ind}_B^G(\chi))$ is k has characteristic different from p ?

(4) Suppose that k has characteristic p and $n = 2$. What is the dimension of $H^1(U, \mathrm{ind}_B^G(\chi))$ over k ?

(5) Suppose that $q = p$ and $n = 2$. How many isomorphisms classes of extensions of k by $\mathrm{ind}_B^G(\chi)$ as a representation of U are there ? Describe them explicitly.

Problem 13 (p-adic numbers)**.** Let p be a prime number. For any $n \geq 1$, let $A_n := \mathbb{Z}/p^n\mathbb{Z}$ and $\pi_n : \mathbb{Z} \to A_n$ the natural projection. Recall that it is a morphism of rings. It factors through a natural morphism of rings $\pi_{n+1,n} : A_{n+1} \to A_n$. Let $\mathcal{A} = \prod_{n \geq 1} A_n$. It is naturally a ring and is naturally endowed with a compact topology. Let
$$i : \mathbb{Z} \to \mathcal{A}, \quad z \mapsto (\pi_n(z))_{n \geq 1}.$$

(1) Verify that i is an injective morphism of rings. Denote by $\mathbb{Z}_p$ the subset of all elements a in $\mathcal{A}$ such that
$$\pi_{n+1,n}(\pi_{n+1}(a)) = \pi_n(a) \quad \forall n \geq 1.$$

(2) Show that $\mathbb{Z}_p$ is a compact topological subring of $\mathcal{A}$ containing $\mathbb{Z} \cong i(\mathbb{Z})$.

(3) Show that the map $\pi_n : \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ extends naturally to a morphism of rings $\pi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$.

(4) Recall what is a neighborhood basis of a point is the product space $\mathcal{A}$. Show that for $a \in \mathbb{Z}_p$, the set of all $(V_k(a))_{k \geq 1}$ is neighborhood basis of a in $\mathbb{Z}_p$, where
$$V_k(a) = \{x \in \mathbb{Z}_p, \quad \pi_k(x) = \pi_k(a)\}.$$

(5) Show that $\mathbb{Z}$ (identified with $i(\mathbb{Z})$) is dense in $\mathbb{Z}_p$.

(6) We represent the elements of A_n by integers between 0 and $p^n - 1$. Let $x \in \mathbb{Z}_p$ with the corresponding sequence $(a_n)_{n \geq 1}$. There is a sequence $(b_i)_{i \geq 0}$ with $0 < b_i < p$ such that
$$a_n = b_0 + b_1 p + \cdots + b_{n-1} p^{n-1}$$
for any $n \geq 1$. Show that the series
$$\sum_{n \geq 0} b_n p^n$$
converges towards x in $\mathbb{Z}_p$.

(7) Show that for $x \in \mathbb{Z}_p$, there is a unique sequence $(b_i)_{i\geq 0}$ with $0 < b_i < p$ such that

$$\sum_{n\geq 0} a_n p^n$$

converges towards x in $\mathbb{Z}_p$.

(8) Let $x = \sum_{n=0}^{\infty} b_n p^n \in \mathbb{Z}_p$, $x \neq 0$. Let

$$val(x) := \mathrm{Inf}\{n, b_n \neq 0\}.$$

We choose $val(0) := +\infty$. This defines map $val : \mathbb{Z}_p \to \mathbb{N} \cup \{+\infty\}$ called the p-adic valuation. Let $x, y \in \mathbb{Z}_p$.

   (a) Show that $val(x) = +\infty$ iff $x = 0$.
   (b) Show that $val(xy) = val(x) + val(y)$.
   (c) Show that $val(x+y) \geq \mathrm{Inf}\{val(x), val(y)\}$. Give the condition for this inequality to be an equality.

(9) Show that $\mathbb{Z}_p$ is an integral domain.

(10) Let $\mathbb{Z}_p^\times$ denote the set of invertible elements in $\mathbb{Z}_p$. Show that $x \in \mathbb{Z}_p^\times$ iff $val(x) = 0$ iff $\pi_1(x) \neq 0$.

(11) Show that $\mathbb{Z}_p$ has a unique maximal ideal and that this ideal is principal. What are its generators ?

(12) Show that $\mathbb{Z}_p$ is a principal ring. Describe a basis of neighborhoods of 0 in terms of its ideals.

(13) Denote by $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$. Check that it contains $\mathbb{Q}$ and that $val : \mathbb{Z}_p \to \mathbb{N} \cup \{+\infty\}$ to map $\mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$ that we still call val. Show that a nonzero element x in $\mathbb{Q}_p$ can be written uniquely in the form $p^{val(x)}y$ where $y \in \mathbb{Z}_p^\times$.

(14) For $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$, ler

$$V_k(a) := \{x \in \mathbb{Q}_p,\ val(x-a) \geq k\}.$$

We consider the topology on $\mathbb{Q}_p$ such that $(V_k(a))_{k\in\mathbb{Z}}$ is a neighborhood basis of a. Show that the topology it induces on $\mathbb{Z}_p$ coincides with the one we defined above. Show that $\mathbb{Q}_p$ is a Hausdorff space. Show that $\mathbb{Q}_p$ is locally compact, that is to say that every point in $\mathbb{Q}_p$ as a compact neighborhood.

(15) Show that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

(16) An absolute value on a field K is a map $K \to \mathbb{R}^+$, $x \mapsto |x|$ satisfying :, for $x, y \in K$:

   (a) $|x| = 0 \Leftrightarrow x = 0$,
   (b) $|xy| = |x|\,|y|$,
   (c) $|x+y| \leq |x| + |y|$.

To such an absolute value we attach the distance $d(x,y) := |x-y|$ on K and this endows K with a topology.

Show that the map $x \mapsto |x| := p^{-val(x)}$ is an aboslute value on $\mathbb{Q}_p$ and that it is nonarchimedean that it to say that it satisfies furthermore :

   (d) $|x+y| \leq \mathrm{Max}\{|x|,\ |y|\}$.

(17) Show that a ball in the metric space $\mathbb{Q}_p$ is open and compact. What is the 1-radius ball ? When do two balls have a nonempty intersection ?

(18) Show that $\mathbb{Q}_p$ is totally disconnected. What is the connected component of a point ?

(19) Show that $\mathbb{Q}_p$ is complete.