We recall that for a vector field $\mathbf{F}$ in $\mathbb{R}^3$, we have:

$$\mathbf{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

$$\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

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1. True or False. Circle the right answer. You do not need to justify your answer. No partial credit.

2 marks (a) Let $\mathbf{F} = (P, Q, R)$ be a vector field in $\mathbb{R}^3$ and assume that P, Q and R have continuous second order partial derivatives. Then $\text{div}(\mathbf{curl}(\mathbf{F})) = 0$.

True False

2 marks (b) The line integral of the vector field $\langle yz, xz, xy \rangle$ along the triangle with vertices $(1, 0, 0)$, $(1, -1, 0)$ and $(1, 1, 0)$, traversed in that order, is equal to 0.

True False

2 marks (c) There exists a vector field $\mathbf{G}$ in $\mathbb{R}^3$ such that $\mathbf{curl}(\mathbf{G}) = \langle x + yz, y + zx, z + xy \rangle$.

True False

2 marks (d) If the curve C is oriented positively and bounds the surface S in the $x - y$ plane, then $\frac{1}{2} \oint_C x dx + y dy$ is equal to the area of S.

True False

2 marks (e) The line integral of the vector field $\frac{1}{x^2+y^2}\langle -y, x \rangle$ along the circle $x^2 + y^2 = 1$ oriented counterclockwise is zero.

True False

Explanations for question 1:

(a) This is because if P, Q, R have continuous 2^{nd} order partial derivatives, then

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} R = \frac{\partial}{\partial y}\frac{\partial}{\partial x} R \quad \text{etc} \ldots$$

(Compute $\operatorname{div}(\overrightarrow{\operatorname{curl} F})$ and you'll see then that it is zero

(b) this vector field is conservative since one can check that

$$\overline{0} = \overline{\text{curl}} < yz, xz, xy >.$$

(and the vector field is defined on $\mathbb{R}^3$)

So the line integral of this vector field along a closed curve is zero.

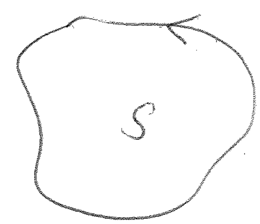
(c) $\text{div} < x+yz, y+zx, z+xy >$

$= 1+1+1 = 3 \neq 0.$

By (a) there is no such $\overline{G}$.

(d) By Green's theorem

$$\frac{1}{2}\int_C \underbrace{x}_{P}\,dx + \underbrace{y}_{Q}\,dy$$



$$= \frac{1}{2}\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = 0.$$

(e) $r(t) = (\cos t, \sin t)$ is a parametrization of the circle with $0 \le t \le 2\pi$

$$\overline{F} := < \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} >.$$

$$\int_C \overline{F}\cdot\overline{dr} = \int_{t=0}^{2\pi} \overline{F}(\overline{r}(t))\cdot\overline{r'(t)}\,dt = \int_{t=0}^{2\pi} \begin{pmatrix} \frac{-\sin t}{1} \\ \frac{\cos t}{1} \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt$$

$$= \int_{t=0}^{2\pi} 1\,dt = 2\pi.$$

Here we notice that if we set

$$P = \frac{-y}{x^2+y^2} \qquad Q = \frac{x}{x^2+y^2}$$

we do have $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$

So it seems that the vector field is conservative. But it is not defined on a simply connected domain: it is not defined at $(0,0)$ therefore we cannot claim that the vector field is conservative (nor can we apply Green's theorem since $(0,0)$ is inside of the circle ...)

2. Let $\mathbf{F} = \langle 3, x, y^2 \rangle$ and S be the surface $z = x^2 + y^2$ with $0 < z < 4$.

$\boxed{\text{1 mark}}$ (a) Compute $\mathbf{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$.

$\boxed{\text{2 marks}}$ (b) Give a parametrization of the surface S. Do not forget the domain of definition.

$\boxed{\text{3 marks}}$ (c) Compute the flux of $\mathbf{curl}(\mathbf{F})$ through S with the upward orientation for S that is to say the double integral

$$\iint_S \mathbf{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

$\boxed{\text{2 marks}}$ (d) Consider the 2-radius circle C centered at $(0, 0, 4)$ and contained in the plane $z = 4$ traversed counterclockwise when seen from above. What is the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$? Justify your answer.

$\boxed{\text{2 marks}}$ (e) Let S' be the disk $x^2 + y^2 < 4$ with $z = 4$. We choose the downward orientation for S'. What is the flux of $\mathbf{curl}(\mathbf{F})$ through S' ? Justify your answer.

(a) $\overrightarrow{\text{curl}}\,\vec{F} = \nabla \times \vec{F} = \langle 2y, 0, 1 \rangle$

(b) $r(u,v) = (u, v, u^2+v^2)$

with $u^2+v^2 < 4$

(c) Note that $\vec{r}_u \times \vec{r}_v = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix} = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}$

points $\underline{\text{upward}}$. So

$$\iint_S \overrightarrow{\text{curl}}\,\vec{F} \cdot \vec{dS} = \iint_{u^2+v^2<4} \overrightarrow{\text{curl}}\,\vec{F}(r(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du\, dv$$

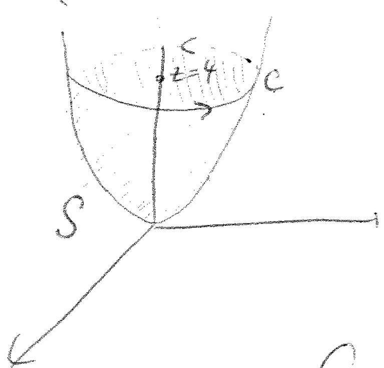
$$= \iint_{u^2+v^2<4} \begin{pmatrix} 2v \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix} du\, dv$$

$$= \iint_{u^2+v^2<4} -4uv + 1 \, du\, dv = 4\pi - 4 \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3 \cos\theta \sin\theta \, dr\, d\theta$$

$$\iint_S \overrightarrow{\text{curl}\, F} \cdot \overrightarrow{dS} = 4\pi - 4 \int_{\theta=0}^{2\pi} \frac{2^4}{4} \sin\theta \cos\theta \, d\theta$$

$$= 4\pi .$$

(d)



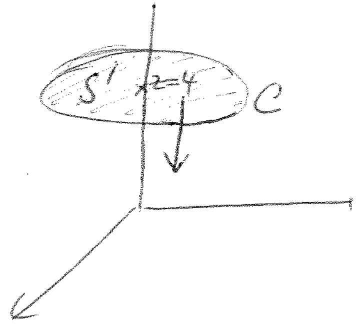
The orientation of S induced by the orientation of C is the upward orientation.

(like in question (c))

So by Stokes' theorem

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S \overrightarrow{\text{curl}\, F} \cdot \overrightarrow{dS} = 4\pi$$

(e)



The orientation of the circle C induced by the downward orientation of the disk S' is clockwise.

So by Stokes theorem

$$\iint_{S'} \overrightarrow{\text{curl}\, F} \cdot \overrightarrow{dS} = -4\pi .$$

3. We consider the plane with equation $x + 2y + z = 5$ and a simple closed curve C on this plane. Suppose that C is oriented counterclockwise when viewed from above. We call $\mathcal{A}$ the area of the surface enclosed by C. Let $\mathbf{F} = \langle \frac{z^2}{4}, \frac{x^2}{2}, y^2 \rangle$

$\boxed{\text{7 marks}}$ (a) Give the expression of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a function of $\mathcal{A}$.

$\boxed{\text{3 marks}}$ (b) Compute the line integral $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ where C_1 is the the curve consisting of the three line segments, L_1 from $(0, 0, 5)$ to $(0, \frac{5}{2}, 0)$, then L_2 from $(0, \frac{5}{2}, 0)$ to $(5, 0, 0)$, and finally L_3 from $(5, 0, 0)$ to $(0, 0, 5)$.

(a) Parametrization of the surface S enclosed by C

$$\vec{r}(u, v) = (u, v, 5 - u - 2v) \quad \text{with domain } D.$$

We know that

$$\mathcal{A} = \iint_S 1 \, dS = \iint_D 1 \, |r_u \times r_v| \, du \, dv.$$

We are going to apply Stokes' theorem to $\vec{F}$, noticing that the orientation of the surface S induced by the chosen orientation of C is upward.

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S \overrightarrow{\text{curl} F} \cdot \vec{dS}$$

$$\overrightarrow{\text{curl} F} = \left(2y, \frac{z}{2}, x\right)$$

$$\vec{r_u} \times \vec{r_v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{points upwards}$$

and has magnitude $\sqrt{6}$. So $\mathcal{A} = \sqrt{6} \iint_D 1 \, du\,dv$

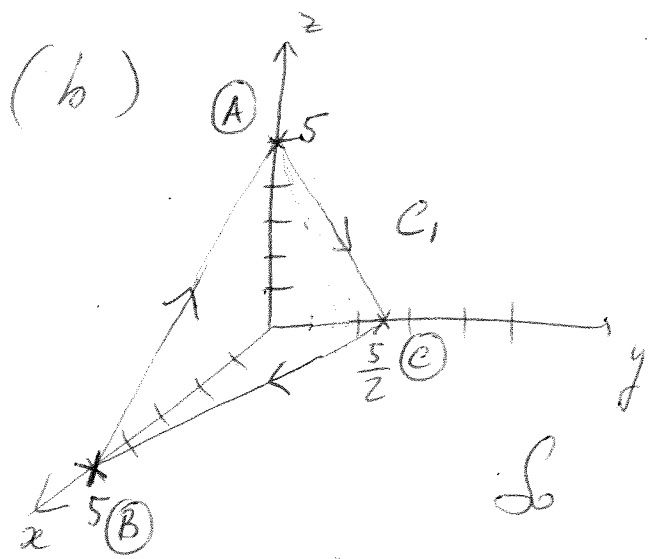
Furthermore :

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \overrightarrow{\text{curl} F}\cdot\vec{dS} = \iint_D \begin{pmatrix} 2v \\ \frac{5-u-2v}{2} \\ u \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} du\,dv$$

$$= \iint_D (2v + 5 - u - 2v + u)\, du\,dv$$

$$= 5 \iint_D 1\, du\,dv = \frac{5A}{\sqrt{6}}$$

(b)



Note that the orientation of the closed curve corresponds to the downward orientation of the enclosed surfa[ce]

So by (a)

$$\int_{C_1} \vec{F}\cdot\vec{dr} = -\frac{5A}{\sqrt{6}} = -\frac{5^3}{4}$$

The area of the triangle ABC is $\frac{|\vec{AB}\times\vec{AC}|}{2}$

$$\vec{AB}\times\vec{AC} = \begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix}\times\begin{pmatrix} 0 \\ 5/2 \\ -5 \end{pmatrix} = \begin{pmatrix} 25/2 \\ 25 \\ 25/2 \end{pmatrix} \quad \text{so } A = \frac{5^2}{2}\sqrt{\frac{3}{2}}$$