$\boxed{\text{7 marks}}$ 1. (a) Calculate the work done by the force field $\mathbf{F} = (-ay, bx, -cz)$ on a particle that moves on a path C, where C is composed of

i. a straight line from $(0,-1,0)$ to $(0,1,0)$,

ii. straight lines from $(0,-1,0)$ to $(1,-1,0)$ to $(0,1,0)$.

$v_1 \; (1,-2,0)$

$\frac{x-1}{1} = \frac{y+1}{-2}$

$\boxed{\text{3 marks}}$ (b) What is the condition on a, b, and c for $\mathbf{F}$ to be conservative ? Hint: you may use part (a).

$\boxed{\text{3 marks}}$ (c) For $a = 2$, $b = -2$ and $c = 1$, evaluate the line integral of $\mathbf{F}$ along the curve C given by $\mathbf{r}(t) = (\ln(1+t^5), e^{t^7}, t)$ with $0 \le t \le 1$.

(a) (i) $r(t) = (0, 2t-1, 0) \qquad 0 \le t \le 1$

$r'(t) = (0, 2, 0)$

$F(r(t)) = (-a(2t-1), 0, 0)$

$$\Rightarrow \int_C F\, dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = 0$$

(ii) $\tilde{C} = C_1 + C_2$

$\Rightarrow C_1 : \; r_1(t) = (t, -1, 0) \Rightarrow F(r_1(t)) = (a, bt, 0), \quad 0 \le t \le 1$

$C_2 : \; r_2(t) = (1-t, 2t-1, 0) \Rightarrow F(r_2(t)) = (-a(2t-1), b(1-t), 0)$

$0 \le t \le 1$

$$\Rightarrow \int_{\tilde{C}} F\, dr = \int_{C_1 + C_2} F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 (a, bt, 0) \cdot (1, 0, 0) + \int_0^1 (-2at+a, -bt+b, 0) \cdot (-1, +2, 0)\, dt$$

$$= \int_0^1 a\, dt + \int_0^1 2at - a - 2bt + 2b\, dt$$

$$= a + \left(at^2 - at - bt^2 + 2bt\right)\Big|_0^1$$

$$= a + \cancel{a} - \cancel{a} - b + 2b$$

$$= a + b$$

①

b) In order to have a conservative vector field, F must be independent of path i.e. for any two paths including C and $\tilde{C}$ (part a) with same end points, we have

$$0 = \int_C F \cdot dr = \int_{\tilde{C}} F \cdot dr = a+b$$

$\Rightarrow$ $a+b=0$ is a necessary condition for F to be conservative.
(NOT sufficient)

Now assuming $a+b=0$ we can find a function f such that

$F = \nabla f$.

$f_x = -ay \Rightarrow f(x,y,z) = -axy + g(y,z)$

$f_y = bx = -ax \Rightarrow f_y = -ax + g_y = bx$

$\Rightarrow g_y = (\underbrace{a+b}_{0})x = 0 \Rightarrow g = g(z)$

$f_z = -cz \Rightarrow g'(z) = -cz \Rightarrow g(z) = -\frac{c}{2}z^2 + K$ constant

$\Rightarrow f(x,y,z) = -axy - \frac{c}{2}z^2 + K$

and $\nabla f = F \Rightarrow a+b=0$ implies that F is conservative and No condition is required on c.

Or alternatively by $a+b=0$, if $F = \langle P, Q, R \rangle$ we have

$P_y = -a = b = Q_x$

$P_z = 0 = R_x \qquad \Rightarrow a+b=0$ is necessary and

$Q_z = 0 = R_y \qquad$ sufficient for F to be conservative.

Because the field is defined on R^3 which is simply connected.

(c) $a = 2 = -b \overset{\text{part (b)}}{\Longrightarrow}$ F conservative i.e. $F = \nabla f$

Thus for $0 \leq t \leq 1$;

$$\int_C F \cdot dr = f(r(1)) - f(r(0))$$
$$= f(\ln 2, e, 1) - f(0, 1, 0)$$

Finding the potential :

$$\left.\begin{array}{l} f_x = -2y \\ f_y = -2x \end{array}\right\} \Rightarrow f(x,y,z) = -2xy + g(z)$$

$$f_z = -z \qquad \Rightarrow f_z = g'(z) = -z \Rightarrow g(z) = -z^2/2$$

$$\Rightarrow \quad f(x,y,z) = -2xy - \tfrac{1}{2} z^2$$

$$\Rightarrow \int_C F \cdot dr = -2e\ln 2 - \tfrac{1}{2}$$

= f(r(1))-f(r(0)) by the fundamental theorem for vector fields.

7 marks 2. Compute the line integral of the vector field

$$\mathbf{F}(x,y) = (ye^{xy}, xe^{xy} + x)$$

along the curve $x(t) = \cos(t)$, $y(t) = 4\sin(t)$ where $0 \le t \le 2\pi$. Hint: decompose $\mathbf{F}$ as a sum of two vectors fields, and justify that one of them is conservative.

$$F(x,y) = \underbrace{(\overbrace{ye^{xy}}^{P}, \overbrace{xe^{xy}}^{Q})}_{F_1} + \underbrace{(0, x)}_{F_2}$$

F_1 is conservative since $\begin{cases} P_y = e^{xy} + xye^{xy} = Q_x \\ \text{and domain } F = \mathbb{R}^2 \text{ simply connected} \end{cases}$

On the other hand;

C: $r(t) = (\cos t, 4\sin t)$, $0 \le t \le 2\pi$ $\Rightarrow r(0) = r(2\pi) \Rightarrow$ closed curve

$$\Rightarrow \int_C F_1 \, dr = 0$$

$$\int_C F_2 \, dr = \int_0^{2\pi} (0, \cos t) \cdot (-\sin t, 4\cos t) \, dt$$

$$= \int_0^{2\pi} 4\cos^2 t \, dt$$

$$= \int_0^{2\pi} 2(1 + \cos 2t) \, dt$$

$$= 4\pi + \underbrace{\sin 2t \Big|_0^{2\pi}}_{0}$$

$$\boxed{= 4\pi} \quad \Rightarrow \int_C F \, dr = 4\pi$$

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$\boxed{\text{7 marks}}$ 3. Calculate the arc length of the curve parameterized by

$$x(t) = \frac{1}{2}t + \frac{1}{2}\cos t \sin t\,, \quad y(t) = \frac{1}{2}\sin^2 t\,; \qquad 0 \le t \le \pi\,.$$

Suggestion: you may use the formulas $\sin(2t) = 2\sin(t)\cos(t)$ and then $1 + \cos(2t) = 2\cos^2(t)$ to simplify the calculations.

this typo had been edited before the midterm date.

$$r(t) = \left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\,,\ \tfrac{1}{2}\sin^2 t\right)$$

$$r'(t) = \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\,,\ \tfrac{1}{2}\sin 2t\right)$$

$$L = \int_0^{\pi} |r'(t)|\,dt = \int_0^{\pi} \sqrt{\tfrac{1}{4} + \underline{\tfrac{1}{4}\cos^2 2t} + \tfrac{1}{2}\cos 2t + \underline{\tfrac{1}{4}\sin^2 2t}}\;dt$$

$$= \int_0^{\pi} \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos 2t}\;dt$$

using the identity $$= \int_0^{\pi} \sqrt{\tfrac{1}{2}(2\cos^2 t)}\;dt$$

$$= \int_0^{\pi} |\cos t|\;dt$$

$$= \int_0^{\pi/2} \cos t\;dt + \int_{\pi/2}^{\pi} -\cos t\;dt$$

$$= \sin t\,\Big|_0^{\pi/2} - \sin t\,\Big|_{\pi/2}^{\pi}$$

$$= 1 + 1 \quad \boxed{= 2}$$

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4. Consider the curve C given by $\mathbf{r}(t) = e^{-t}\cos(t)\mathbf{i} + e^{-t}\sin(t)\mathbf{j} + \sqrt{2}e^{-t}\mathbf{k}$.

3 marks (a) Let $L(t)$ denote the arc length of the curve from the point $(1, 0, \sqrt{2})$ to the point with parameter t. Compute $\lim_{t\to+\infty} L(t)$.

1 mark (b) Compute the unit tangent vector $\mathbf{T}(t)$.

1 mark (c) Compute the unit normal vector $\mathbf{N}(t)$ and check that $\mathbf{N}(0) = (-\sqrt{2}/2, -\sqrt{2}/2, 0)$.

3 marks (d) Give the coordinates of the center of the osculating circle at parameter $t = 0$. We recall that the curvature at parameter t is given by $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ and that the osculating circle at parameter t has radius $1/\kappa(t)$.

4 marks (e) Give the equation of the osculation plane at parameter $t = 0$.

1 mark (f) Is C a plane curve?

$$r'(t) = (-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin t + e^{-t}\cos t,\ -\sqrt{2}e^{-t})$$

point $(1, 0, \sqrt{2})$ corresponds to $t = 0$

$$|r'(t)| = \Big[e^{-2t}\cos^2 t + e^{-2t}\sin^2 t + 2e^{-2t}\cancel{\sin t\cos t}$$
$$+ e^{-2t}\sin^2 t + e^{-2t}\cos^2 t - 2e^{-2t}\cancel{\sin t\cos t}$$
$$+ 2e^{-2t}\Big]^{1/2}$$
$$= (4e^{-2t})^{1/2} = 2e^{-t}$$

(a)

$$L(t) = \int_0^t |r'(s)|\,ds = \int_0^t 2e^{-s}\,ds = -2e^{-s}\Big|_0^t$$
$$= -2e^{-t} + 2$$

$$\Rightarrow \lim_{t\to+\infty} L(t) = 2$$

(b) $$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{2e^{-t}}\left(-e^{-t}(\cos t + \sin t),\ e^{-t}(\cos t - \sin t),\ -\sqrt{2}\, e^{-t}\right)$$

$$= \frac{1}{2}\left(-\cos t - \sin t,\ \cos t - \sin t,\ -\sqrt{2}\right)$$

(c) $$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$T'(t) = \tfrac{1}{2}\left(+\sin t - \cos t,\ -\sin t - \cos t,\ 0\right)$$

$$|T'(t)| = \tfrac{1}{2}\left(\underline{\sin^2 t + \cos^2 t} - \cancel{2\sin t}\cos t + \underline{\sin^2 t + \cos^2 t} + \cancel{2\sin t}\cos t\right)^{1/2}$$

$$= \sqrt{2}/2$$

$$\Rightarrow N(t) = \frac{1}{\sqrt{2}}\left(\sin t - \cos t,\ -\sin t - \cos t,\ 0\right)$$

$$\Rightarrow N(0) = 1/\sqrt{2}\,(-1,\ -1,\ 0) = (-\sqrt{2}/2,\ -\sqrt{2}/2,\ 0)$$

(d) center : $P + RN$

$$K(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\sqrt{2}/2}{2e^{-t}} = \sqrt{2}/4\ e^{t}$$

$$\xrightarrow{t=0} K(0) = \sqrt{2}/4 \Rightarrow R = 4/\sqrt{2}$$

$$r(0) = (1,\ 0,\ \sqrt{2}) = P \Rightarrow \text{Center}: (1, 0, \sqrt{2}) + \frac{4}{\sqrt{2}}(-\sqrt{2}/2,\ -\sqrt{2}/2,\ 0)$$

$$= (-1,\ -2,\ \sqrt{2})$$

(e) $B(t) = T(t) \times N(t)$

$$\Rightarrow \quad B(0) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{\sqrt{2}}{2}\right) \times \left(-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}, 0\right)$$

$$= \left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{\sqrt{2}}{2}\right)$$

$$P(1, 0, \sqrt{2}) \Rightarrow -\tfrac{1}{2}(x-1) + \tfrac{1}{2}y + \tfrac{\sqrt{2}}{2}(z - \sqrt{2}) = 0$$

$$\Rightarrow \quad -x + y + \sqrt{2}z = 1$$

(f) For C to NOT be a plane curve, we need to show that there exists a "t" such that

$$-e^{-t}\cos t + e^{-t}\sin t + 2e^{-t} \neq 1$$

~~But when $t=0$, the above equality holds;~~

$$\cancel{-1 + 2 = 1 \checkmark}$$

If C was a plane curve that is to say a curve contained in a plane P, then at every parameter t the osculating plane to the curve would be equal to the plane P.

In the previous question, we computed the osculating plane to the curve at parameter t=0. If the curve was a plane curve, it would be contained in the plane with equation $-x + y + \sqrt{2} z = 1$. To prove that the curve is not a plane curve, it is enough to find a parameter t such that the corresponding point of the curve is not on this plane: we look for t such that

$-e^{-t} \cos(t) + e^{-t} \sin(t) + 2 e^{-t} \neq 1$

We "check" (admittedly you may need a calculator or the knowledge of the value of e to really check it, but you got full credit for thinking of something like this) for example: at $t = \pi$: $3e^{-\pi} \neq 1$.