1. TRUE or FALSE:

   (a) If A is an $n \times n$ matrix with nonzero determinant and $AB = AC$ then $B = C$. TRUE

   (b) A square matrix with zero diagonal entries is never invertible. FALSE

   (c) A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is one-to-one if and only if its standard matrix has nonzero determinant. TRUE

   (d) Every spanning subset of $\mathbb{R}^4$ contains a basis for $\mathbb{R}^4$. TRUE

   (e) A linearly independent subset of $\mathbb{R}^n$ has at most n elements. TRUE

   (f) Every subspace of $\mathbb{R}^3$ contains infinitely many vectors. FALSE

   (g) The system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A. TRUE

2. Indicate if each of the following is a linear subspace:

   (a) The set of all vectors parallel to a fixed vector $\mathbf{v}$. SUBSPACE

   (b) All the vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with $x_1 + x_3 = 1$. NOT A SUBSPACE
   (The zero vector does not belong to the above set!)

   (c) The intersection of two subspaces of $\mathbb{R}^n$. SUBSPACE

   (d) The set of vectors in $\mathbb{R}^3$ with two equal components. NOT A SUBSPACE
   (The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ both belong to this set but $2\mathbf{e}_1 + \mathbf{e}_2$ does not.)

3. Determine if each of the following matrices is invertible? If not, explain why. If so, compute its inverse.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 3 & 0 & -4 \\ 3 & -2 & -2 & 8 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -4 & 0 & 1 & 0 & 0 \\ 3 & -2 & -2 & 8 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & -3 & 2/3 & 1 & 0 \\ 0 & 0 & 1 & -2/3 & -1 & -1/3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & -2/3 & -3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right)$$

The last row is zero which implies the matrix is not invertible.

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -5 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1 & 5/3 & 1 & -1/3 & 0 \\ 0 & 0 & -1/3 & -1 & 2/3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1 & 5/3 & 1 & -1/3 & 0 \\ 0 & 0 & 1 & 3 & -2 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -4 & 3 & 5 \\ 0 & 0 & 1 & 3 & -2 & -3 \end{array}\right)$$

This implies the original matrix is invertible and the inverse is:

$$\begin{pmatrix} -1 & 1 & 1 \\ -4 & 3 & 5 \\ 3 & -2 & -3 \end{pmatrix}$$

4. Compute the determinant of the following matrices:

$$\begin{pmatrix} 0 & 2 & 3 \\ -1 & -1 & 4 \\ 2 & -2 & 2 \end{pmatrix}$$

We expand with respect to the first column:

$$-(-1)\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 10 + 22 = 32.$$

$$\begin{pmatrix} 0 & 0 & 5 & 5 \\ 0 & 2 & 6 & 12 \\ 1 & 4 & 7 & 12 \\ 2 & 8 & 14 & 15 \end{pmatrix}$$

After subtracting twice the third row from the fourth, we get the following matrix

$$\begin{pmatrix} 0 & 0 & 5 & 5 \\ 0 & 2 & 6 & 12 \\ 1 & 4 & 7 & 12 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

which has the same determinant. After expanding with respect to the last row, we can see that the determinant is equal to

$$-9\begin{vmatrix} 0 & 0 & 5 \\ 0 & 2 & 6 \\ 1 & 4 & 7 \end{vmatrix}$$

We expand with respect to the first column for the 3×3 and the determinant can be computed by:

$$(-9)\begin{vmatrix} 0 & 5 \\ 2 & 6 \end{vmatrix} = 90.$$

5. Suppose $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ is an $n \times n$ matrix and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A. For each of the following statements say if they are true or false and justify your answer.

   (a) If $\det(A) = 0$ then the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is linearly dependent. TRUE

   Solution: Recall that if $\det(A) = 0$ then the system of equations $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. But $A\mathbf{x}$ is a just a linear combination of the columns of A. So a nontrivial solution of $A\mathbf{x} = \mathbf{0}$ means a nonzero linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is zero and therefore $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is linearly dependent.

   (b) If $\det(A) = 0$ then $\mathbf{a}_n$ is a linear combination of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$. FALSE

   Solution: Not true in general because for example we can choose $\mathbf{a}_1 = \cdots = \mathbf{a}_{n-1} = \mathbf{0}$ and $\mathbf{a}_n$ nonzero. Then $\det(A) = 0$ but $\mathbf{a}_n$ is not a linear combination of the other vectors.

   (c) If $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly independent and $\det(A) = 0$ then $\mathbf{a}_n$ is a linear combination of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$. TRUE

   Solution: By part (a) above we know that $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly dependent. So there exist scalars $c_1, \ldots, c_n$, not all equal to zero, so that $c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$. We claim

$c_n \neq 0$, otherwise $c_1\mathbf{a}_1 + \cdots + c_{n-1}\mathbf{a}_{n-1} = \mathbf{0}$ and scalars $c_1, \ldots, c_{n-1}$ are not all equal to zero. This contradicts the assumption that $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly independent. So $c_n \neq 0$ and we can see:

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0} \Rightarrow c_n\mathbf{a}_n = -c_1\mathbf{a}_1 - \cdots - c_{n-1}\mathbf{a}_{n-1}$$

and therefore

$$\mathbf{a}_n = -(c_1/c_n)\mathbf{a}_1 - \cdots - (c_{n-1}/c_n)\mathbf{a}_{n-1}.$$

(d) If the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution then the determinant of the matrix $B = (\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\, \mathbf{b})$ obtained from replacing the last column by $\mathbf{b}$ is zero. FALSE

Solution: Not necessarily. For example let $A = I_n$ be the identity matrix. Then it is easy to see that $A\mathbf{x} = \mathbf{e}_n$ has a solution, where $\mathbf{e}_n$ is the last column of I_n. But replacing the last column by $\mathbf{e}_n$ does not change the matrix and the determinant remains equal to 1.

(e) There exists a vector $\mathbf{c}$ which is not in the span of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$. TRUE

Solution: We have seen that at least n vectors are required to span $\mathbb{R}^n$.

(f) If $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$ is linearly independent then there exists a vector $\mathbf{c}$ such that the determinant of the matrix $C = (\mathbf{a}_1 \cdots \mathbf{a}_{n-1}\, \mathbf{c})$ obtained from replacing the last column by $\mathbf{c}$ has nonzero determinant. TRUE

Solution: By part (e), there is a vector $\mathbf{c}$ which is not a linear combination of $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$. Then by (c), $\det(A) \neq 0$.

6. Given a set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ in $\mathbb{R}^n$, if A is the matrix $A = (\mathbf{a}_1 \ \ldots \ \mathbf{a}_n)$, we write

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \det(A)$$

(a) Suppose that $\det(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) = 2$ for a set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ in $\mathbb{R}^4$. Find:

$$\det(\mathbf{w} + 2\mathbf{v}, \mathbf{v}, \mathbf{z}, 3\mathbf{u})$$

Solution: We notice that the matrix $(\mathbf{w} + 2\mathbf{v}, \mathbf{v}, \mathbf{z}, 3\mathbf{u})$ is obtained from the matrix $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z})$ by the following column operations:

$$(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) \to (3\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) \to (3\mathbf{u}, \mathbf{v}, \mathbf{w}+2\mathbf{v}, \mathbf{z}) \to (\mathbf{w}+2\mathbf{v}, \mathbf{v}, 3\mathbf{u}, \mathbf{z}) \to (\mathbf{w}+2\mathbf{v}, \mathbf{v}, \mathbf{z}, 3\mathbf{u})$$

The first operation, multiplies a column by 3, the second adds twice a column to another and the third and fourth interchange two columns. Recall that the first one multiplies the determinant by 3, the second does not change the determinant and the third and fourth just change the sign. This shows that

$$\det(\mathbf{w} + 2\mathbf{v}, \mathbf{v}, \mathbf{z}, 3\mathbf{u}) = 3(-1)(-1)\det(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}) = 6.$$