Name: Answer Key

Student Number: ____________________

Math 221: Matrix Algebra
Midterm 2

(1) Indicate whether each of the following hold by *writing the complete word* True *or* False (you will lose points for simply writing T or F). *You do not need to justify your answers.*

(a) If A and B are $n \times n$ matrices, and $AB = 0$, then either $A = 0$ or $B = 0$. False

(b) Every $n \times n$ invertible matrix is row equivalent to the $n \times n$ identity matrix. True

(c) Every matrix is the standard matrix of a linear transformation. True

(d) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$. False

(e) If a set S of vectors spans $\mathbb{R}^n$, then S is linearly dependent. False

(f) If A is not invertible, then A^T is not invertible. True

(2) Indicate whether or not each of the following is a linear subspace by writing either *yes*, if it *is* a linear subspace, and otherwise *no*. *You do not need to justify your answers.*

(a) $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + x_2 - x_3 = 0 \right\}$. yes

(b) The union of two subspaces of $\mathbb{R}^n$. no

(c) The set $\{\mathbf{0}\}$ consisting of the zero vector in $\mathbb{R}^n$. yes

(d) The set of vectors in $\mathbb{R}^4$ whose third coordinate is 1. no

(3) Answer each of the following:

(a) Compute the determinant of the following matrix A:

$$A = \begin{pmatrix} 1 & -1 & 2 & 1/3 \\ 0 & 0 & 1 & -5 \\ -1 & 2 & 1 & 7 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

expanding along the last row:

$$\det(A) = 3 \begin{vmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = (-1)(2-1) = -1$$

note that this is the matrix A in Part (b).

(b) Is the following matrix B invertible? If not, explain why. If so, compute its inverse.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

from a computation in (a), we saw that $\det(A) \neq 0$, so A is invertible.

To compute A^{-1}, we row reduce

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -5 & 1 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

Therefore the inverse of A is

$$A^{-1} = \begin{pmatrix} 2 & -5 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(4) Let $A = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5)$ be the 4×5 matrix whose columns are the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, and $\mathbf{a}_5$ in $\mathbb{R}^4$. Suppose the reduced echelon form of A is

$$\mathrm{REF}(A) = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) What is the rank of A? 3

(b) What is the dimension of the nullspace of A? 2

(c) Give a basis for the column space of A. $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$

(d) Give a basis for the nullspace of A.

$x_1 = x_2 + x_4 \quad x_3 = x_4 \quad x_5 = 0$

$$Null(A) = \left\{ \begin{pmatrix} x_2+x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

Therefore $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $Null(A)$.