Math 221: Matrix Algebra
Midterm I - Sample Questions - January, 2013

1. TRUE or FALSE

   (a) A homogenous system with more variables than equations has a nonzero solution.

   (b) If $\mathbf{x}$ is a solution to the system of equations $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b}$ then $2\mathbf{x}$ is also a solution to the same system.

   (c) If A has a zero column then the homogenous system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

   (d) The only linear transformation which is both one-to-one and onto is the identity map.

2. Find the complete solution set to the system:

$$\begin{cases} 2x_1 + 2x_2 + x_3 + x_4 - x_5 = 6 \\ 2x_1 + 2x_2 + x_3 + 2x_4 - 3x_5 = 10 \\ -2x_1 - 2x_2 + 2x_3 - 3x_4 + 6x_5 = -12 \end{cases}$$

3. Find the reduced row echelon matrix which is row equivalent to

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 2 & 8 & 14 & 10 \\ 1 & 3 & 5 & 12 \end{pmatrix}$$

4. Suppose

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Find all the solutions to the system of equations

$$A\mathbf{x} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

   (b) Express the vector

$$\begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$$

   as a linear combination of columns of A.

5. In each of the following cases, determine whether the given vector is in the set spanned by the columns of the given matrix:

(a) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ with $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ with $\begin{pmatrix} 3 & 2 & -4 \\ 1 & -1 & -3 \\ 1 & 5 & 3 \end{pmatrix}$

6. Determine if the set $S = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}$ is linearly independent in $\mathbb{R}^3$ and explain why.

7. Determine if each of the following functions is a linear transformation. If it is the case find the matrix representing the transformation with respect to the standard bases.

   (a) $L : \mathbb{R}^3 \to \mathbb{R}^1$, with $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = -x_2 - x_1$.

   (b) $L : \mathbb{R}^2 \to \mathbb{R}^2$, with $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$.

   (c) $L : \mathbb{R}^3 \to \mathbb{R}$ with $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = 1$.

   (d) $L : \mathbb{R}^2 \to \mathbb{R}^2$ with $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_2 - x_1 \\ 0 \end{pmatrix}$.

8. For the following linear transformations, find the standard matrix and also determine if they are one-to-one or onto.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}$ with $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = x_2$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(\mathbf{e}_1) = \mathbf{e}_2 + \mathbf{e}_3$ and $T(\mathbf{e}_2) = -\mathbf{e}_1 + \mathbf{e}_2$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 + x_2 \\ -x_1 - x_2 \end{pmatrix}$.

9. Suppose the following vectors in $\mathbb{R}^3$ are given

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

   (a) Determine if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   (b) Determine if S spans $\mathbb{R}^3$.

(c) Express the vector

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

as a linear combination of elements of the vectors in S.

10. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is not onto. Answer the following questions and explain your answers.

    (a) What is the size of the standard matrix for T?

    (b) How many pivots does the standard matrix of T has?

    (c) Can T be one-to-one?