Worksheet 7

We recall that given a vector $\mathbf{u}$ and the line L spanned by $\mathbf{u}$, the orthogonal projection of a vector $\mathbf{x}$ onto L is

$$\text{proj}_L(\mathbf{x}) = \frac{\mathbf{x}.\mathbf{u}}{\|\mathbf{u}\|^2}\ \mathbf{u}. \tag{1}$$

Now, instead of a line L, consider more generally a subspace W of $\mathbb{R}^n$. We are interested in the orthogonal projection onto W. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is an **orthogonal** basis for W, then for any $\mathbf{x} \in \mathbb{R}^n$

$$\text{proj}_W(\mathbf{x}) = \frac{\mathbf{x}.\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\ \mathbf{u}_1 + \cdots + \frac{\mathbf{x}.\mathbf{u}_m}{\|\mathbf{u}_m\|^2}\ \mathbf{u}_m. \tag{2}$$

Given a point A of the space $\mathbb{R}^n$, the distance from A to the subspace W is the length

$$\|\overrightarrow{OA} - \text{proj}_W(\overrightarrow{OA})\|$$

of the vector $\overrightarrow{OA} - \text{proj}_W(\overrightarrow{OA})$.

Drawing :

Problem 1. Let $\mathbf{u} = \begin{pmatrix}1\\2\end{pmatrix}$ and L the line spanned by $\mathbf{u}$. Compute the distance to L from the points $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$.

Problem 2. Let L be the 1-dimensional subspace of $\mathbb{R}^4$ spanned by $\mathbf{u} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$. We consider

$$\text{proj}_L : \mathbb{R}^4 \to \mathbb{R}^4$$

the orthogonal projection on L.

(A) What is the range of proj_L ?

(B) Let A be the matrix of proj_L in the standard basis. What is the rank of A ?

(C) Compute the coordinates of the images by proj_L of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$.

(D) Compute A.

(E) Give a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $L^\perp = \{\mathbf{x} \in \mathbb{R}^4,\ \mathbf{x}.\mathbf{u} = 0\}$.

(F) Consider the matrix P whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{u}$. What is $P^{-1}AP$?

(G) Compute A^{100}.

Problem 3. In $\mathbb{R}^3$, consider the plane P with equation $x + z = 0$.

(A) Find an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for P (that is to say an orthogonal set that is a basis for P).

(B) Find a basis $\{\mathbf{u}_3\}$ for $P^\perp$.

(C) Consider the orthogonal projection $T = \mathrm{proj}_{P^\perp} : \mathbb{R}^3 \to \mathbb{R}^3$. It is a linear transformation.

 (a) What is the range of T ?

 (b) Give the matrix B of T in the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

 (c) Compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, $T(\mathbf{e}_3)$.

 (d) Give the matrix A of T in the standard basis.

 (e) What is the matrix Q such that $B = Q^{-1}AQ$?

(D) (optional) Consider the orthogonal projection $S = \mathrm{proj}_P$ on P.

 (a) What is $S + T$?

 (b) What is the matrix C of S in the standard basis ?

 (c) Can you retrieve your answer to the previous question using Equation (2) at the beginning of this worksheet ?

(E) What is A^2 ? What is C^2 ?

Problem 4. Let α be a positive number and L the line in $\mathbb{R}^2$ with equation $y = \alpha x$. We consider the (orthogonal) reflection T about L. Let θ be the angle from the vector $\mathbf{e}_1$ to the vector $\mathbf{u}_1 := \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$. We want to determine the matrix A of T in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(A) First approach (very optional). Let proj_L be the orthogonal projection on L and $\mathrm{proj}_{L^\perp}$ the orthogonal projection on $L^\perp$.

 (a) Give the matrix M of proj_L in the standard basis.

 (b) Give a basis $\{\mathbf{u}_2\}$ for $L^\perp$ and the matrix N of $\mathrm{proj}_{L^\perp}$ in the standard basis.

 (c) What is $\mathrm{proj}_L - \mathrm{proj}_{L^\perp}$?

 (d) Give the matrix A of T in the standard basis in terms of α.

(B) Second approach.

 (a) Let $\mathbf{u}_2 := \begin{pmatrix} -\alpha \\ 1 \end{pmatrix}$. After checking that $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, give the matrix D of T in $\mathcal{B}$.

 (b) Consider the matrix P whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$. What is the link between A and D ?

 (c) Compute the matrix A in terms of α.

(C) Give the coordinates of the image of $\mathbf{e}_1$ by T in terms of θ.

(D) (very optional) What is $\tan(\theta)$ in terms of α ? Prove the following identities :

$$\cos(2\theta) = \frac{1 - \tan^2(\theta)}{1 + \tan^2(\theta)}, \quad \sin(2\theta) = \frac{2\tan(\theta)}{1 + \tan^2(\theta)}.$$

 NB : one can prove these equalities also by direct computation and using other trigonometric identities.

Problem 1 $\quad L = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\} \quad \vec{u} = \begin{pmatrix}1\\2\end{pmatrix} \quad \|\vec{u}\|^2 = 5$

$$d(A, L) = \|\overrightarrow{OA} - \text{proj}_L(\overrightarrow{OA})\| = \left\|\overrightarrow{OA} - \frac{\overrightarrow{OA}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right\|$$

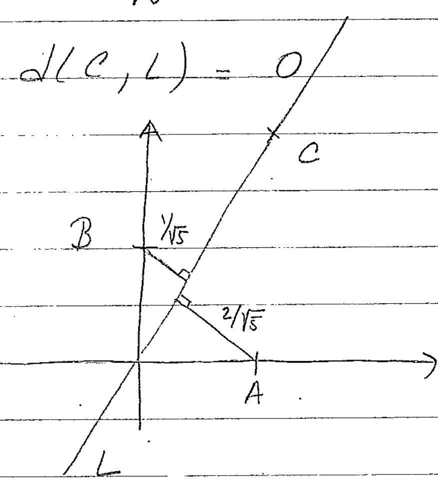
$A = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{proj}_L(\overrightarrow{OA}) = \frac{1}{5}\begin{pmatrix}1\\2\end{pmatrix} \quad d(A, L) = \left\|\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1/5\\2/5\end{pmatrix}\right\| = \left\|\begin{pmatrix}4/5\\-2/5\end{pmatrix}\right\|$

$$= \frac{1}{5}\sqrt{16+4} = \frac{2}{\sqrt{5}}$$

$B = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{proj}_L(\overrightarrow{OB}) = \frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix} \quad d(B, L) = \left\|\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}2/5\\4/5\end{pmatrix}\right\| = \left\|\begin{pmatrix}-2/5\\1/5\end{pmatrix}\right\|$

$$= \frac{1}{\sqrt{5}}$$

$C = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{proj}_L(\overrightarrow{OC}) = \overrightarrow{OC} \Longrightarrow d(C, L) = 0$



In the worksheet I gave
I changed B into $\begin{pmatrix}0\\3\end{pmatrix} = D$

$d(D, L) = \left\|\begin{pmatrix}0\\3\end{pmatrix} - \frac{6}{5}\begin{pmatrix}1\\2\end{pmatrix}\right\| = \ldots$

Problem 2 $\quad \vec{u} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad L = \text{Span}(\vec{u})$

(A) $\text{Range}(\text{proj}_L) = L$

(B) The rank of A is the dimension of the range of proj_L (by definition of the rank of a matrix)
so $\text{rank}(A) = 1$

(C) $\text{proj}_L(\vec{e_i}) = \dfrac{\vec{e_i}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \dfrac{1}{4}\vec{u} = \begin{pmatrix}1/4\\1/4\\1/4\\1/4\end{pmatrix}$

(D) $A = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ (We see again that it has rank 1)

(E) $L^\perp = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ such that } x_1 + x_2 + x_3 + x_4 = 0 \right\}$

A basis for $L^\perp$ is $\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{v_1}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{\vec{v_2}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v_3}}$

(F) By definition, $P^{-1}AP$ is the matrix of the transformation proj_L in the basis $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{u}\}$

But $\mathrm{proj}_L(\vec{v_1}) = \mathrm{proj}_L(\vec{v_2}) = \mathrm{proj}_L(\vec{v_3}) = \vec{0}$

$\mathrm{proj}_L(\vec{u}) = \vec{u}$

So $P^{-1}AP = \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{D}$

(You can double check without computing P^{-1} by checking $AP = PD$

Here $P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$)

(G) $A = PDP^{-1}$

$A^{100} = PD^{100}P^{-1} = PDP^{-1} = A$ Since $D^2 = D$

So $D^{100} = D$.

Problem 3: (A) $\overline{u_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\overline{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(B) $\overline{u_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(C) $T = \mathrm{proj}_{P^\perp}$ (a) $\mathrm{Range}(T) = P^\perp$

(b) $T(\overline{u_1}) = \overline{0}$

$T(\overline{u_2}) = \overline{0}$ so $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$T(\overline{u_3}) = \overline{u_3}$

(c) $T(\overline{e_1}) = \dfrac{\overline{e_1} \cdot \overline{u_3}}{\|\overline{u_3}\|^2} \overline{u_3} = \dfrac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$T(\overline{e_2}) = \dfrac{\overline{e_2} \cdot \overline{u_3}}{\|\overline{u_3}\|^2} \overline{u_3} = \overline{0}$

$T(\overline{e_3}) = \dfrac{\overline{e_3} \cdot \overline{u_3}}{\|\overline{u_3}\|^2} \overline{u_3} = \dfrac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

(e) P is the matrix with colums $\{\overline{u_1}, \overline{u_2}, \overline{u_3}\}$

$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ (To double check that $B = P^{-1}AP$ just verify $PB = AP$)

(D) $S = \mathrm{proj}_P$. (a) $S + T = id_{\mathbb{R}^3}$

(b) $C + A = I \implies C = I - A$

$= \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$

(c) We can retrieve this result by computing
$S(\vec{e_1})$ $S(\vec{e_2})$ $S(\vec{e_3})$

Since $\{\vec{u_1}, \vec{u_2}\}$ is an orthogonal basis for P, we have

$$S(\vec{e_i}) = \frac{\vec{e_i} \cdot \vec{u_1}}{\|\vec{u_1}\|^2}\vec{u_1} + \frac{\vec{e_i} \cdot \vec{u_2}}{\|\vec{u_2}\|}\vec{u_2}$$

Recall $\vec{u_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\vec{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$S(\vec{e_1}) = \frac{1}{2}\vec{u_1} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$S(\vec{e_2}) = \vec{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

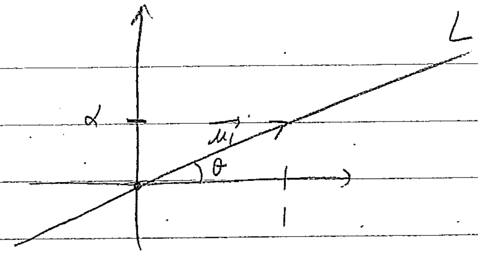
$$S(\vec{e_3}) = \frac{-1}{2}\vec{u_1} = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix}$$

So $C = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$

(f) A^2 is the matrix of $T \circ T$ in the standard basis. ~~Since~~ Since $T \circ T = T$ we have $A^2 = A$

Likewise $C^2 = C$.

Problem 4



$$\vec{u_1} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

(A) (a) $$\text{proj}_L(\vec{e_1}) = \frac{\vec{e_1} \cdot \vec{u_1}}{\|\vec{u_1}\|^2}\,\vec{u_1} = \frac{1}{1+\alpha^2}\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

$$\text{proj}_L(\vec{e_2}) = \frac{\vec{e_2} \cdot \vec{u_1}}{\|\vec{u_1}\|^2}\,u_1 = \frac{\alpha}{1+\alpha^2}\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

$$M = \begin{pmatrix} 1/1+\alpha^2 & \alpha/1+\alpha^2 \\ \alpha/1+\alpha^2 & \alpha^2/1+\alpha^2 \end{pmatrix}$$

(b) $L^\perp$ = Span of $\left\{\begin{pmatrix} -\alpha \\ 1 \end{pmatrix}\right\}$ $\quad \vec{u_2} = \begin{pmatrix} -\alpha \\ 1 \end{pmatrix}$

$$\text{proj}_L(\vec{e_1}) = \frac{\vec{e_1} \cdot \vec{u_2}}{\|\vec{u_2}\|^2}\,\vec{u_2} = \frac{-\alpha}{1+\alpha^2}\begin{pmatrix} -\alpha \\ 1 \end{pmatrix}$$

$$\text{proj}_L(\vec{e_2}) = \frac{\vec{e_2} \cdot \vec{u_2}}{\|\vec{u_2}\|^2}\,\vec{u_2} = \frac{1}{1+\alpha^2}\begin{pmatrix} -\alpha \\ 1 \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{\alpha^2}{1+\alpha^2} & -\alpha/1+\alpha^2 \\ -\alpha/1+\alpha^2 & 1/1+\alpha^2 \end{pmatrix}$$

(c) $\text{proj}_L - \text{proj}_{L^\perp} = T$

(d) so $M - N = A$ and

$$A = \begin{pmatrix} \frac{1-\alpha^2}{1+\alpha^2} & \frac{2\alpha}{1+\alpha^2} \\ \frac{2\alpha}{1+\alpha^2} & -\frac{1-\alpha^2}{1+\alpha^2} \end{pmatrix}$$

(B) (a) $T(\vec{u_1}) = \vec{u_1}$

$T(\vec{u_2}) = -\vec{u_2}$ so $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

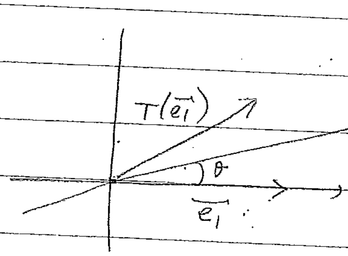
(b) $P = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$ $D = P^{-1} A P$

(c) $A = P D P^{-1}$ $P^{-1} = \begin{pmatrix} \frac{1}{1+\alpha^2} & \frac{\alpha}{1+\alpha^2} \\ \frac{-\alpha}{1+\alpha^2} & \frac{1}{1+\alpha^2} \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/1+\alpha^2 & \alpha/1+\alpha^2 \\ -\alpha/1+\alpha^2 & 1/1+\alpha^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ \alpha & -1 \end{pmatrix} \quad \underbrace{\begin{pmatrix} \frac{1-\alpha^2}{1+\alpha^2} & \frac{2\alpha}{1+\alpha^2} \\ \frac{2\alpha}{1+\alpha^2} & -\frac{1-\alpha^2}{1+\alpha^2} \end{pmatrix}}_{A}$$

(C)



$T(\vec{e_1}) = \begin{pmatrix} \cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$

(D) $\tan(\theta) = \alpha$

The first column of A is $T(\vec{e_1})$

So $\cos(2\theta) = \dfrac{1-\alpha^2}{1+\alpha^2} = \dfrac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$.

$\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2\theta}$